

Springer Monographs in Mathematics

Editors-in-chief

Isabelle Gallagher, Paris, France

Minhyong Kim, Oxford, UK

Series editors

Sheldon Axler, San Francisco, USA

Mark Braverman, Toronto, Canada

Maria Chudnovsky, Princeton, USA

Tadahisa Funaki, Tokyo, Japan

Sinan C. Güntürk, New York, USA

Claude Le Bris, Marne la Vallée, France

Pascal Massart, Orsay, France

Alberto Pinto, Porto, Portugal

Gabriella Pinzari, Napoli, Italy

Ken Ribet, Berkeley, USA

René Schilling, Dresden, Germany

Panagiotis Souganidis, Chicago, USA

Endre Süli, Oxford, UK

Shmuel Weinberger, Chicago, USA

Boris Zilber, Oxford, UK

This series publishes advanced monographs giving well-written presentations of the “state-of-the-art” in fields of mathematical research that have acquired the maturity needed for such a treatment. They are sufficiently self-contained to be accessible to more than just the intimate specialists of the subject, and sufficiently comprehensive to remain valuable references for many years. Besides the current state of knowledge in its field, an SMM volume should ideally describe its relevance to and interaction with neighbouring fields of mathematics, and give pointers to future directions of research.

More information about this series at <http://www.springer.com/series/3733>

Władysław Narkiewicz

The Story of Algebraic Numbers in the First Half of the 20th Century

From Hilbert to Tate

 Springer

Władysław Narkiewicz
University of Wrocław
Wrocław, Poland

ISSN 1439-7382 ISSN 2196-9922 (electronic)
Springer Monographs in Mathematics
ISBN 978-3-030-03753-6 ISBN 978-3-030-03754-3 (eBook)
<https://doi.org/10.1007/978-3-030-03754-3>

Library of Congress Control Number: 2018960727

Mathematics Subject Classification (2010): 11Rxx, 11-03, 01A60

© Springer Nature Switzerland AG 2018

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

To the memory of my wife

Preface

The aim of this book is to give a survey of results in the theory of algebraic numbers achieved in the first half of the twentieth century and may be viewed as a companion to my previous book *Rational Number Theory in the 20th Century* in which the part of number theory dealing with rational numbers has been treated. It is an attempt to fulfil the wish of H. S. Vandiver expressed in 1960 in his paper [4185], and perhaps it might be helpful in preventing rediscoveries.

Chapter 1 gives a concise presentation of the beginnings of the theory of algebraic numbers. One finds here first a description of the work on special cases of algebraic integers done by Gauss, Dirichlet and Eisenstein, followed by Kummer's work on cyclotomic fields. Then the creation of the general theory by Kronecker and Dirichlet is treated, and the chapter concludes with a short description of the related work of other mathematicians, including Hermite, Minkowski, Frobenius and Stickelberger.

In Chap. 2 one finds a presentation of the work of Hilbert, who in his report on algebraic numbers summarized the state of their theory at the end of the nineteenth century, as well of Hensel, who created p -adic and \mathfrak{p} -adic numbers, which turned out to be an indispensable tool in future research. In the last part of the chapter the first steps towards creation of the class-field theory, characterizing Abelian extensions of algebraic number fields, are described.

Chapter 3 covers the first twenty years of the twentieth century. In the first section we present its central subject, the use of analytic methods in the theory of algebraic numbers. This has been initiated by Landau, who established the Prime Ideal Theorem giving asymptotics for the number of prime ideal with bounded norms. The next big achievement was Hecke's proof of the continuation of the Dedekind zeta-function to a meromorphic function on the plane, and the study of several generalizations of Dirichlet L -functions to number fields, developed by him and Landau. The second section presents the results dealing with the algebraic structure, and the last section is devoted to other results achieved in the beginning of the twentieth century.

The central themes of Chap. 4 are the creation of the modern ideal theory by Emmy Noether, and the establishment of fundamental theorems of class-field theory by Takagi and Artin. Also various other questions were considered at that time, for example the first results in the additive theory of algebraic numbers obtained by Rademacher.

In Chap. 5 we present first the progress in the study of the structure of number fields, the central subject being the existence of normal and normal integral bases, and then consider some additive questions, mainly on sums of squares. The next section concentrates on the simplification of the class-field theory by Hasse and Chevalley, and the following sections concern i.a. the class-number and class-group of quadratic fields, the question of the existence of the Euclidean algorithm, the distribution of algebraic integers on the complex plane and infinite extensions of number fields. Chapter 6 covers The Forties, the main results being obtained by Brauer and Siegel.

In all chapters one will find also some selected information about the subsequent developments of the arising problems.

I am very grateful to my friends Kálmán Győry and Andrzej Schinzel for reading the draft of the book and providing several comments and suggestions. I thank also the referees of the book for several important hints.

I am very grateful to the Springer staff for helpful cooperation in preparing the publication. My special thanks go to Ms Elena Griniari and Ms Angela Schulze-Thomin.

Wrocław, Poland

Władysław Narkiewicz

Contents

1	The Birth of Algebraic Number Theory	1
1.1	The Beginning	1
1.1.1	Euler	1
1.1.2	Gauss	2
1.1.3	Dirichlet	12
1.2	First Steps	16
1.2.1	Eisenstein	16
1.2.2	Kummer	19
1.3	Establishing the Theory	31
1.3.1	Kronecker	31
1.3.2	Geometrical Approach: Hermite and Minkowski	36
1.3.3	Dedekind	43
1.3.4	Frobenius and Stickelberger	53
1.4	Other Results	55
1.5	Remarks	60
2	The Turn of the Century	63
2.1	David Hilbert	63
2.1.1	First Results	63
2.1.2	Zahlbericht	65
2.1.3	After the Zahlbericht	71
2.2	Kurt Hensel	73
2.2.1	Field Index and Monogenic Fields	73
2.2.2	Discriminants	77
2.2.3	p -Adic Numbers	79
2.3	The Beginnings of Class-Field Theory	85
2.3.1	Kronecker's Jugendtraum	85

2.3.2	Heinrich Weber	88
2.3.3	Hilbert's Class-Field	91
3	First Years of the Century	95
3.1	Analytic Methods	95
3.1.1	Edmund Landau	95
3.1.2	Erich Hecke and the New L -Functions	103
3.2	Structure	114
3.2.1	Steinitz	114
3.2.2	Galois Groups	117
3.2.3	Discriminants and Integral Bases	120
3.2.4	Units	122
3.2.5	Splitting Primes	125
3.2.6	Reciprocity	126
3.3	Class-Number	127
3.3.1	Quadratic Fields	127
3.3.2	Cyclotomic Fields	130
3.4	Other Questions	133
3.5	Books	138
4	The Twenties	141
4.1	Structure	141
4.1.1	Ideal Theory	141
4.1.2	Integral Bases, Discriminants, Factorizations	144
4.1.3	Units	150
4.2	Analytical Methods	153
4.2.1	Quadratic Reciprocity Law	153
4.2.2	Sums of Powers	155
4.2.3	Sums of Primes	157
4.2.4	Piltz Problem	158
4.2.5	Values of Zeta-Functions	159
4.3	Class-Field Theory	161
4.3.1	Takagi	161
4.3.2	Artin	164
4.3.3	Hasse	172
4.4	Class-Number and Class-Group	175
4.4.1	Quadratic Fields	175
4.4.2	Other Fields	178
4.5	Other Questions	180
4.5.1	Galois Groups	180
4.5.2	Algebraic Numbers in the Plane	182
4.5.3	Infinite Extensions	184
4.5.4	Varia	185
4.5.5	Books	186

- 5 The Thirties** 189
 - 5.1 Structure 189
 - 5.1.1 Ideal Theory 189
 - 5.1.2 Integral Bases, Discriminants, Factorizations 190
 - 5.1.3 Units 194
 - 5.2 Class-Field Theory 197
 - 5.2.1 Hasse 197
 - 5.2.2 Chevalley 202
 - 5.3 Class-Number and Class-Group 205
 - 5.3.1 Quadratic Fields 205
 - 5.3.2 Other Fields 211
 - 5.4 Other Questions 212
 - 5.4.1 Additive Problems 212
 - 5.4.2 Galois Groups 214
 - 5.4.3 Euclidean Algorithm 215
 - 5.4.4 Algebraic Numbers on the Plane 215
 - 5.4.5 Infinite Extensions 222
 - 5.4.6 Local Fields 223
 - 5.4.7 Algebraic Numbers and Matrices 225
 - 5.4.8 Varia 226
- 6 The Forties** 229
 - 6.1 Analytic Methods 229
 - 6.1.1 General Results 229
 - 6.1.2 Additive Problems 234
 - 6.2 The Class-Number 237
 - 6.2.1 Class-Number of Quadratic Fields 237
 - 6.2.2 Class-Number of Cyclotomic Fields 242
 - 6.3 Class-Field Theory 245
 - 6.4 Euclidean Algorithm 249
 - 6.5 Other Topics 251
 - 6.6 Books 257
- Bibliography** 259
- Author Index** 417
- Subject Index** 435