Undergraduate Texts in Mathematics

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A Readable Introduction to Real Mathematics

Second Edition

Springer

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To the memory of Harold and Esther Rosenthal who gave us (and others) the gift of mathematics.

Preface to the Second Edition

This second edition is an expanded and improved version of the first. The last two chapters are entirely new. The other chapters have been revised, taking into account the comments of many readers. We are particularly grateful to Florin Catrina, Jonathan Korman, Andrew Nicas, Carolyn Pitchik, Heydar Radjavi and Zack Wolske for their suggestions.

The preface to the first edition has been rewritten and divided into two prefaces, one for readers and one for instructors.

There are undoubtedly further improvements that could be made. We would appreciate your sending any comments, corrections, or suggestions to any of the authors at their e-mail addresses given below.

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Preface for Readers

The fundamental purpose of this book is to teach you to understand mathematical thinking. We have tried to do that in a way that is clear and engaging, and emphasizes the beauty of mathematics. You may be reading this book on your own or as a text for a course you are enrolled in. Regardless of your reason for reading this book, we hope that you will find it understandable and interesting.

This book contains a lot of mathematics. We do not expect you to necessarily read all of it. In the Preface for Instructors, we describe possible courses that use only parts of the book.

Mathematics is a huge and growing body of knowledge; no one can learn more than a fraction of it. But the central thing to learn is how to think mathematically. It is our experience that mathematical thinking can be learned by almost anyone who is willing to make a serious attempt. We invite you to make such an attempt by reading at least part of this book. It is important not to let yourself be discouraged if you can't easily understand something. Everyone learning mathematics finds some concepts baffling at first, but usually, with enough effort, the ideas become clear.

One way in which mathematics gets very complex is by building on itself; some mathematical concepts are built on a foundation of many other concepts and thus require a great deal of background to understand. That is not the case for the topics discussed in this book. Reading this book does not require any background other than basic high school algebra and, for parts of Chapters 9 and 12, some high school trigonometry.

A few questions, among the many, that you will easily be able to answer after reading the relevant parts of this book are the following: Is $13^{217} \cdot 37^{92} \cdot 41^{15} = 19^{111} \cdot 29^{145} \cdot 43^{12} \cdot 47^{5}$ (see Chapter 4)? Is there a largest prime number (i.e., a largest whole number whose only factors are 1 and itself) (Theorem 1.1.5)? If a store sells one kind of product for 9 dollars each and another kind for 16 dollars each and receives 143 dollars for the total sale of both, how many products did the store sell at each price (Example 7.2.7)? How do computers send secret messages to each other (Chapter 6)? How is the size of an infinite set defined? Are there more fractions than there are whole numbers? Are there more real numbers than there are fractions? Is there a smallest infinity? Is there a largest infinity (Chapter 10)? What are complex numbers (Chapter 9)? Is .3333 ... really equal to $\frac{1}{3}$ (Example 13.2.8)? What are some infinite-dimensional spaces (Section 14.5)?

The hardest theorem proven in this book concerns the construction of angles using a compass and a straightedge. (A straightedge is a ruler-like device but without measurements marked on it. Straightedges are used to draw lines connecting two points.) If you are given any angle, it is easy to bisect it (i.e., divide it into two equal subangles) by using a compass and a straightedge (we will show you how to do that). This and many similar results were discovered by the Ancient Greeks. The Ancient Greeks wondered whether angles could be "trisected" in the sense of being divided into three equal subangles using only a straightedge and a compass. A lot of mathematics beyond that conceived of by the Ancient Greeks was required to solve this problem; it was not solved until the nineteenth century. It can be proven that many angles, including an angle of 60 degrees, cannot be so trisected. We present a complete proof of this as an illustration of complicated but beautiful mathematical reasoning.

The most important question you'll be able to answer after reading at least several chapters of this book, although you will have difficulty formulating the answer in words, is: what is mathematical thinking really like? If you read and understand several chapters and do a fair number of the problems that are provided, you will certainly have a feeling for mathematical thinking.

We hope that you read this book carefully. Reading mathematics is not like reading a novel, a newspaper, or anything else. As you go along, you have to really reflect on the mathematical reasoning that is being presented. After reading a description of an idea, think about it. When reading mathematics you should always have a pencil and paper at hand to rework what you read.

The essence of mathematics consists of theorems, which are statements proven to be true. We will prove a number of theorems. When you begin reading about a theorem, think about why it may be true before you read our proof. In fact, at some points you may be able to prove the theorem we state without looking at our proof at all. In any event, you should make at least a small attempt before reading the proof in the book. It is often useful to continue such attempts while in the middle of reading the proof that we present; once we have gotten you a certain way towards the result, see if you can continue on your own.

If you adopt such an approach and are patient, we believe that you will learn to think mathematically. We are also convinced that you will feel that much of the mathematics that you learn is beautiful, in the sense that you will find that the logical argument that establishes the theorem is what mathematicians call "elegant."

We chose the material for this book based on the following criteria: the mathematics is beautiful, it is useful in many mathematical contexts, and it is accessible without much mathematical background. The theorems that we prove have applications to mathematics and to problems in other subjects.

Each chapter ends with a section entitled "Problems." The "Problems" sections are divided into three subsections. You should do some of the "Basic Exercises" to ensure that you have an understanding of the fundamentals of the chapter.

The subsections entitled "Interesting Problems" contain problems whose solutions depend upon the material of the chapter and seem to have mathematical or other interest. The subsections labeled "Challenging Problems" contain problems that we expect you will, indeed, find to be quite challenging. You should not be discouraged if you cannot solve some of the problems. However, if you do solve problems that you find difficult at first, especially those that we have labeled "challenging," we hope and expect that you will experience some of the pleasure and satisfaction that mathematicians feel upon discovering new mathematics.

Each chapter is divided into sections. Important items, such as definitions and theorems, are numbered in a way that locates them within a chapter and a section of that chapter. We put the chapter number, then the section number, and then the number of the item within that section. For example, 7.2.4 refers to the fourth numbered item in section two of chapter seven.

Readers who wish to omit some of the material (perhaps only at first) should be aware of the following. Chapters 1, 2, 4, and 8 may be read without reading any other parts of this book. Chapter 5 depends on Chapters 3 and 4, and Chapter 6 requires Chapter 5. Some of the examples in Chapter 7 depend on Chapter 6; the rest of the chapter is independent of Chapter 6. Chapter 8 uses Chapter 4. Chapters 9, 10, and 11 are essentially independent of each other and of all other chapters. Chapter 12 basically depends only on Chapter 11 and on the concepts of rational and irrational numbers as discussed in Chapter 8. Chapter 13 can be read independently of the other chapters, and Chapter 14 does not require any of the previous material except for the essential properties of convergence of infinite series, as discussed in Chapter 13.

Preface for Instructors

A glance at the table of contents and the Preface for Readers will give you an idea of the material covered in this text.

Some features of this book include:

- Complete proofs that an angle of 60 degrees cannot be trisected with a straightedge and compass (Corollary 12.3.24) and that an angle of an integral number n degrees can be constructed with a straightedge and compass if and only if n is a multiple of 3 (Theorem 12.4.13).
- A thorough discussion of the Principle of Mathematical Induction (Chapter 2).
- A chapter that provides an introduction to Euclidean plane geometry (Chapter 11).
- A complete description of RSA encryption (7.2.5).
- A fairly extensive treatment of cardinality (Chapter 10).
- An introduction to infinite-dimensional spaces (Chapter 14).
- Using the least upper bound property to establish theorems about convergence of infinite series (Section 13.4).
- Showing that real numbers can be represented by infinite decimals (Section 13.6).
- A proof that the infinite series consisting of the reciprocals of the prime numbers diverges (Theorem 13.7.8).

Since the only prerequisite for understanding this book is high school algebra, it is suitable as a textbook for a wide variety of courses. In particular, it is our view that appropriate parts of the text could be used for courses for mathematics or science majors, for courses for other students who want to get an appreciation of mathematics, and for courses for prospective teachers. The book is also written so as to be useable for independent study by anyone who is interested in learning mathematics. In particular, mathematically inclined high school students might be directed to this book.

The main purpose of this book is to teach mathematical thinking. Some instructors like to begin such a course by discussing basic logic and different kinds of proofs. Others prefer to present some interesting mathematics simply and clearly, with the expectation that students will learn to think mathematically by being gently exposed to the mathematics presented.

We are in the latter camp.

The text begins with a basic introduction to the natural numbers. This is followed by a chapter that contains a thorough discussion of mathematical induction. A student who has learned to understand most of the material in that chapter will have obtained some appreciation of mathematical thinking. Learning the material in other parts of the book will deepen the student's understanding and will also teach the student a lot of interesting mathematics. The textbook provides the opportunity for you to choose from a variety of mathematical topics.

The following are some descriptions of different courses for which part or all of this book could serve as a text. There are many other variants that instructors could devise.

A course covering most of the book would take two semesters. Such a course would be suitable for students majoring in mathematics, statistics, computer science, or physics.

On the other hand, there are several different one-semester courses that could be based on parts of the book. Instructors can vary the level of these courses by the pace at which they proceed, the difficulty of the problems that they assign, and the material they omit.

One natural possibility would be to begin at page 1, proceed at whatever pace is comfortable for you and your students, and then see where you end up.

Other possibilities involve omitting some of the chapters. It is our opinion that Chapters 1, 2, 4, and 8 should be part of most courses using this book. Additional chapters can be chosen based on the needs of the students, the interests of the instructor, and the time available. For example, Chapters 3, 5, 6, and 7 could be added (i.e., so that the course covers Chapters 1 through 8). Alternatively, Chapters 10 and 13 and/or 14 might be included.

A course containing a proof that some angles cannot be trisected with straightedge and compass could be based on Chapters 1, 2, 4, 8, 11, and 12. Other chapters could be added if time permits.

Fairly leisurely "mathematics appreciation" courses could cover Chapters 1, 2, 3, 4, and 5, or Chapters 1, 2, 4, 8, and 10, or Chapters 1, 2, 4, 8, and 13, or Chapters 1, 2, 4, 8, and the part of Chapter 14 before Definition 14.5.3.

A one-semester course for prospective or actual teachers of high school mathematics could cover Chapters 1, 2, 4, 8, 11, and 12. It is our view that Chapter 12 should be of substantial interest to teachers. If they are already familiar with the fundamentals of Euclidean geometry as presented in Chapter 11, there would likely be time to add one or more of Chapters 9, 10, 13, or 14. If the instructor does not wish to present Chapter 12, a good course for teachers could be based on Chapters 1 through 8.

Lectures on parts of some of the chapters can give students a taste of the topic. For example, a very brief introduction to cardinality could consist of the part of Chapter 10 up through Theorem 10.2.3. Chapter 14 describes some finite and infinite-dimensional spaces. The part before Definition 14.5.3 is completely independent of the rest of the book; the balance requires the concept of convergence of series.

Chapter 11 is an essentially self-contained introduction to geometry.

Chapter 13, which does not significantly rely on any other chapters, is intended to provide a first introduction to concepts of analysis by explaining convergence of infinite series in a direct manner. The idea of "adding lots of terms to get close to the sum" has some intuitive appeal. In our experience, many students who are taught the traditional approach are confused by the distinction between convergence of the sequence of terms and convergence of the sequence of partial sums. That is why we do not discuss convergence of any sequences other than sequences of partial sums. Also, we do not use "sigma notation" within the chapter since some students find it to be a barrier to understanding. We define least upper bounds and use that concept to rigorously prove the fundamental theorems about convergence. The connections to the more standard approaches are established in the last problem of the chapter.

Using a text that contains more than will be covered in the course you are teaching provides an opportunity to encourage interested students to do some reading on their own, before or after the course ends.

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