Part IV
Local Aspects of Spectral Analysis and the Exponential Representation Problem

This part continues our study of mean periodic functions. Here we indicate how the results of Part III can be applied to questions of spectral analysis and spectral synthesis for translation-invariant subspaces.

A translation-invariant subspace $V \subset \mathcal{E}(\mathbb{R}^n)$ is said to admit *spectral analysis* if V contains an exponential, i.e., if there exists $z \in \mathbb{C}^n$ such that $f(x) = e^{i\langle z,x\rangle_{\mathbb{C}}}$ belongs to V. If the exponential polynomials belonging to V are dense in V, we say that V admits *spectral synthesis*. In case every translation-invariant subspace admits spectral analysis (synthesis), we say that spectral analysis (synthesis) holds in $\mathcal{E}(\mathbb{R}^n)$. The *exponential representation problem* for V consists in the representation of an arbitrary element of V as an integral over the set of exponential polynomials belonging to V.

If V is a translation-invariant subspace of $\mathcal{E}(\mathbb{R}^n)$, then there exists a family $\mathcal{T} \subset \mathcal{E}'(\mathbb{R}^n)$ such that $f \in V$ if and only if $f * T = 0$ for all $T \in \mathcal{T}$. This follows from the Hahn–Banach theorem. Thus, the above problems are reduced to the study of systems of convolution equations of compact support in $\mathbb{R}^n$. The remarks presented below give an indication of the spirit which animates the study of these questions.

1. Schwartz [188] proved that spectral synthesis holds in $\mathcal{E}(\mathbb{R}^1)$.
2. Gurevich [102] discovered that translations were not sufficient to generate a spectral synthesis for dimensions greater than one. More precisely, there exists six distributions $T_1, \dots, T_6 \in \mathcal{E}'(\mathbb{R}^n)$ $(n \geqslant 2)$ such that $\{z \in \mathbb{C}^n : \widehat{T_1}(z) = \dots = \widehat{T_6}(z) = 0\} = \varnothing$ but $\{f \in \mathcal{D}'(\mathbb{R}^n) : f * T_1 = \dots = f * T_6 = 0\} \neq \{0\}$.
3. Brown, Schreiber, and Taylor [42] showed that every translation-invariant rotation-invariant subspace of $\mathcal{E}(\mathbb{R}^n)$, $n \geqslant 2$, is spanned by the polynomial-exponential functions it contains.
4. Different exponential representations for solutions of linear partial differential equations with constant coefficients on $\mathbb{R}^n$ are given by Ehrenpreis [69, Chap. 7], and Palamodov [165, Chap. 6]. There are many works devoted to the exponential representation problem for systems of convolution equations (see, for example, Berenstein and Struppa [19]).
5. The complete reconstruction, or *deconvolution*, of f given $T \in \mathcal{T}$ and $f * T$ is of great interest. The theory of deconvolution has its roots in the work of Wiener [261] and Hörmander [125], and has been developed into a working theory by Berenstein et al. (see [16, 24, 25, 27, 28]).
6. Further developments deal with analogous questions for other spaces. In particular, many authors studied the case of symmetric spaces and the Heisenberg group (see, e.g., Berenstein [11], Berenstein and Gay [15], Berenstein and Zalcman [26], Thangavelu [210], and Wawrzrynczyk [251–255]).

In Part IV we study related problems for systems of convolution equations on domains of homogeneous spaces. The main difference with the above-mentioned results is that we do not have any longer the group of translations at our disposal. The absence of the group structure provides a serious complicating factor.

In Chaps. 18 and 19 we present results for Euclidean spaces. The case of symmetric spaces is investigated in Chaps. 20 and 21. Finally, results for the phase space and the Heisenberg group is briefly discussed in the comments.