Part III
Mean Periodicity

Mean periodic functions are a far-reaching generalization of ordinary periodic functions. The notion of a mean periodic function was introduced by Delsarte [54] and was afterwards developed by Schwartz [188]. According to Schwartz, an infinitely differentiable function f on $\mathbb{R}^n$ is said to be *mean periodic* if the closed subspace $V(f)$ generated by f and all its translates is proper in $\mathcal{E}(\mathbb{R}^n)$. Equivalently, by the Hahn–Banach theorem, $f \in C^\infty(\mathbb{R}^n)\backslash\{0\}$ is mean periodic if and only if there exists a compactly supported distribution $T \neq 0$ such that $f * T = 0$. These equations generalize linear partial differential equations with constant coefficients. If $f * T = 0$, we say that f is mean periodic with respect to T. Analogously, by a mean periodic function on a symmetric space or on the Heisenberg group we mean a solution of convolution equation of compact support.

In the course of the study of mean periodic functions a complicated theory arises, which can be developed rather far. A remarkable feature of this theory is that its results are closely related to a wide variety of problems in contemporary mathematics.

In each case there is a striking difference between the behavior of mean periodic functions. We now illustrate this by some examples.

Example I. Let f be a mean periodic function on $\mathbb{R}^1$. Then $V(f)$ contains an exponential $e^{i\lambda x}$ for some $\lambda \in \mathbb{C}$. This result is due to Schwartz [188]. An exact analogue of the Schwartz theorem fails to be true in the case $\mathbb{R}^n$, $n \geqslant 2$ (see Gurevich [102]).

Example II. Let M be a compact Riemannian manifold and L the Laplace–Beltrami operator on M. If f is a twice differentiable function on M such that $Lf \geqslant 0$ everywhere, then f is a constant function in view of the classical Hopf lemma (see Kobayashi and Nomizu [137], Vol. II, Note 14). This statement shows that a compact symmetric space has no mean periodic functions with respect to $T = L\,\delta_0$ except for constant functions. On the other hand, such functions exist on symmetric spaces of noncompact type (see Sect. 10.3).

Example III. Let μ_r stand for the normalized surface measure on the sphere $S_r = \{z \in \mathbb{C}^n : |z| = r\}$, and let $\varphi_k(z) = e^{-|z|^2/4}\ {}_1F_1(-k, n; |z|^2/2)$, $k \in \mathbb{Z}_+$. If $\varphi_k|_{S_r} = 0$, then it follows from (12.35) and (12.53) that φ_k is mean periodic with respect to μ_r. Note that φ_k is a Schwartz class function. This is in sharp contrast with the case of ordinary mean periodic functions on $\mathbb{R}^n$. As is well known, no mean periodic function on $\mathbb{R}^n$ can be integrable.

In this part we wish to arrive at a better understanding of mean periodic functions on the spaces under consideration. Results related to this theme are numerous and diverse. In our treatment of mean periodic functions we discriminate between the following aspects: (i) group and infinitesimal properties of mean periodic functions; (ii) support theorems; (iii) characterization of uniqueness sets; (iv) multidimensional analogues of the distribution ζ_T; (v) description of various classes of mean periodic functions; (vi) a periodic in the mean extension; (vii) analogues of Liouville's property; (viii) approximation theorems.

Chapters 13 and 14 treat the case of Euclidean spaces. A detailed study of mean periodic functions on symmetric spaces is contained in Chaps. 15 and 16. Chapter 17 is devoted to the case of the phase space and the Heisenberg group.