Part II
Transformations with Generalized Transmutation Property Associated with Eigenfunctions Expansions

This is perhaps the central part of the book from the point of view of the mathematical machinery. In it we develop the theory of transmutation operators as a key tool in the study of mean periodic functions on multidimensional domains.

Transmutation operators arise naturally from eigenfunctions expansions of Laplacians. In this part we will deal with Euclidean spaces, symmetric spaces of noncompact type, compact two-point homogeneous spaces, and the phase space associated to the Heisenberg group—the contexts in which there is already a well-established spectral theory of Laplacians.

In Chap. 6 we give some preliminary results about entire functions and distributions. Chapter 7 provides a brief introduction to the theory of special functions. Although these two chapters can be viewed as auxiliary, some of the results presented here are new.

Chapter 8 deals with a subject that is basically a topic in nonharmonic Fourier series and which, at first glance, may seem to have little to do with our principal concern in Part II, namely transmutation operators. Nevertheless, its main results will be essential later in applications of the theory of transmutation operators to mean periodic functions.

The main development of transmutation operators starts with Chap. 9, where we treat the case $\mathbb{R}^n, \;\; n \geqslant 2$. Chapters 10–12 are devoted to the case of symmetric spaces and the phase space. Here appear three types of expansions: (1) Bessel functions, (2) Jacobi functions (this further subdivides into two subcases), and (3) Laguerre functions. For each type of expansion, we define transmutation operators and investigate the following basic questions: the generalized homomorphism property with respect to suitable convolution algebras, support properties, the homeomorphism property with respect to suitable distribution spaces, explicit inversion formulas, the images of certain special functions, normative type estimates, connections with the dual Abel transform, and applications to positive definite functions. The generalized homomorphism property is the crucial one; it relates the mean periodicity on the spaces in question to that on $\mathbb{R}^1$, and allows many proofs in Parts III and IV to be carried out by reduction to the one-dimensional case.