Part I
Symmetric Spaces. Harmonic Analysis on Spheres

The notion of symmetric space is among the most important notions in differential geometry. They are defined as Riemannian manifolds M with the following property: each $p \in M$ is an isolated fixed point of an involutive isometry s_p of M. There is only one such s_p. It is called the symmetry at the point p.

The theory of symmetric spaces was initiated by É. Cartan in 1926 and was vigorously developed by him in the late 1920s. Symmetric spaces have a transitive group of isometries and can be represented as coset spaces G/K, where G is a connected Lie group with an involutive automorphism σ whose fixed-point set is (essentially) K. This property was used by É. Cartan to classify them.

In Chap. 1 we review general notions and facts related to the theory of symmetric spaces which will be used throughout the book. We restrict ourselves to only a minimum of auxiliary information. Standard definitions in Riemannian geometry and Lie group theory are assumed; references for them are given, when necessary.

In Chaps. 2 and 3 symmetric spaces of rank one play a privileged role. Their geometrical structure is so rich that one can characterize them in several other ways. In particular, they, together with the Euclidean spaces $\mathbb{R}^n$ ($n = 1, 2, \dots$), comprise the two-point homogeneous spaces. These are the Riemannian manifolds M with the property that for any two pairs points (p_1, p_2) and (q_1, q_2) satisfying $d(p_1, p_2) = d(q_1, q_2)$, where d is the distance on M, there exists an isometry mapping p_1 to q_1 and p_2 to q_2.

The rank one symmetric spaces of noncompact type are the real, complex, and quaternionic hyperbolic spaces $\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$, $\mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$, and $\mathrm{Sp}(n,1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ and the Cayley hyperbolic plane $F_4^*/\mathrm{Spin}(9)$. In a dual manner, the compact symmetric spaces of rank one are the various projective spaces corresponding to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{C}a$ and the Euclidean spheres. Hyperbolic spaces can be identified with the unit ball in Euclidean space. We give this identification in Chap. 2. Euclidean spaces can be regarded as parts of projective spaces. Chapter 3 contains a detailed discussion of this imbedding. We have attempted to render calculations in the text as explicit as possible. This circumstance is the source of very extensive information about symmetric spaces of rank one.

Roughly speaking, the group K acts transitively on the unit sphere S in the tangent space $\mathfrak{p}$ to a symmetric space G/K of rank one. The purpose of Chap. 4 is an explicit description of the decomposition of $L^2(S)$ into irreducible components under the action of K. Since K in its action on S is contained in the orthogonal group, we see that $\mathcal{H}^k$, the space of homogeneous kth-degree harmonic polynomials on $\mathfrak{p}$, is invariant under the action of K. Identifying elements of $\mathcal{H}^k$ with their restrictions to S, we have $L^2(S) = \sum_{k=0}^{\infty} \mathcal{H}^k$. In this way the question reduces to the problem of decomposing each $\mathcal{H}^k$ according to the action of K. To solve it we use the realizations of G/K obtained in Chaps. 2 and 3.

In Chap. 5 we consider K-finite eigenfunctions of the Laplace–Beltrami operator on rank one symmetric spaces G/K of noncompact type. Integral representations for these functions give us non-Euclidean analogues of the plane waves $e^{i\lambda\langle x,\eta\rangle_{\mathbb{R}}}$, $x \in \mathbb{R}^n$. They play an important role in harmonic analysis on G/K.

The results of Part I are used in the sequel to study mean periodic functions.