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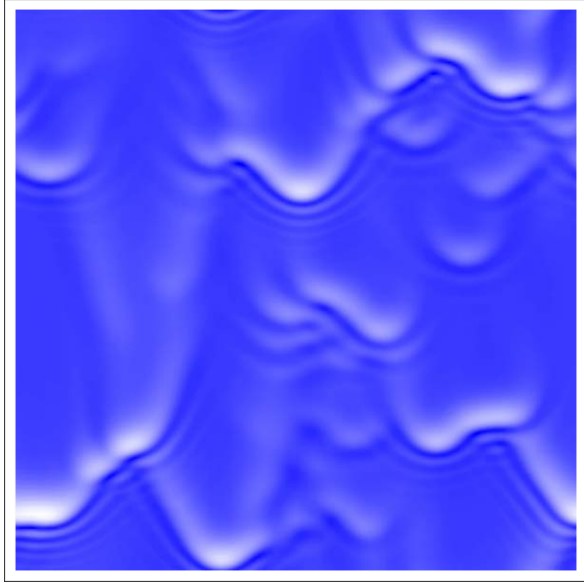
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# Falling Liquid Films

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# Preface

The wavy dynamics on a liquid film flow down an inclined plate is an everyday life phenomenon, easily observable on windows or on sloped pavements in the midst of a rainfall. It is a fascinating sight and so the design of many fountains includes falling liquid films to captivate and entertain passers-by. From the scientific point of view, such flows are part of the general class of free-boundary problems, which hold a strategic position both in pure and applied sciences. The occurrence of free-boundaries and interfaces, i.e., material or geometric frontiers between regimes with different physical properties not a priori prescribed, arises in disparate in nature, inherently nonlinear problems, from fluid and solid mechanics and combustion to financial mathematics, material science and glaciology. Not surprisingly therefore, the wavy dynamics of falling liquid films has attracted not only the attention of Sunday strollers but also of many researchers, and for several decades, to the point that literally hundreds, if not thousands of research papers have been devoted to this topic. Falling films have also been the subject of several books and monographs (see, e.g., [3, 44]) as well as reviews (see, e.g., [201]).

Such considerable interest, which continues up to date with several new developments, stems not only from the inner beauty of the phenomenon but also from its many technological applications, in particular in relation to chemical engineering processes. Typical examples include evaporators and related heat exchange processes with heat transport from a hot wall to a film and vapor condensation, absorbers/mass exchange processes with absorption of dilute gas and coating processes for which the hydrodynamic behavior of the initial liquid film can affect the quality of the final coated surface. The cooling of microelectronic equipment or the separation of multi-component mixtures in the chemical and food industries are often ensured by means of falling films. Falling films even represent the state-of-the-art technique in the sugar industry and constitute the basic components in sea-water desalination plants. Film heat exchangers are commonly used as condensers of cooling agents in cryogenic technology. In addition, films are also used as lubricant layers for the flow of crude oil in pipes and channels, or as the means of thermal protection of the combustion chamber walls in the design of rocket engines.

As far as the use of falling films for heat/mass transport applications is concerned, besides small thermal resistance and large contact area at small specific flow

rates, another advantage is a drastic enhancement of heat/mass transport [58]. For example, Frisk & Davis [97] and Goren & Mani [106] have shown that heat/mass transport across a wavy film can increase by as much as 10–100% compared to flat films. Therefore liquid film flows play a central role in the development of efficient means for interfacial heat/mass transfer in engineering applications.

With regards to fundamental research efforts, it is not just the presence of a free boundary that contributes to the complexity of film flows, but also many other challenging aspects, e.g., heating effects and the way they influence the film flow, three-dimensional effects and chemical reactions, all with many different subtleties and peculiarities that have not been fully resolved as of yet. It is precisely for this reason that falling film flows are still the subject of active research, with several new developments as noted above.

In the recent past, the study of the transition from a state of order to one of disorder in spatially extended systems through low-dimensional dynamical models has been one of the many routes taken by physicists in their quest to understand the development of “spatio-temporal chaos” (or “low-dimensional turbulence”) and even the onset of usual turbulence. A transition to spatio-temporal chaos also characterizes the dynamics of a falling liquid film. More specifically, one observes a well-organized cascade of bifurcations that leads from the flat film state (a “laminar” state) to a state of disorder/spatio-temporal chaos even at low Reynolds numbers. In the latter state, although the film surface appears to be random one can still identify robust “coherent structures,” which continuously interact with each other. Such structures are described well with techniques from nonlinear dynamics and dynamical systems theory. The falling liquid film also shares many analogies and features with other open flow hydrodynamic systems, such as developing boundary layers.

However, unlike many open-flow hydrodynamic systems, the long wave nature of the instabilities observed on a falling film and the low-to-moderate values of the Reynolds number render the problem amenable to a thorough theoretical and numerical investigation within the framework of the long-wave theory. As the waves are long compared to the film thickness, or equivalently, deformations of the free surface are weak, the viscosity of the fluid ensures a great coherence of the flow across the film. These fortuitous characteristics inherent to the falling film problem, enable us to drastically reduce the complexity of the basic equations and to obtain systems of simplified model equations. The advantage of these models is to isolate the underlying physical mechanisms of the phenomena associated with the nonlinear wave evolution on a falling film and to simulate them extensively at a reduced analytical and numerical cost. Hence, a falling liquid film can serve as a canonical reference system for the study of the general problem of transition to spatio-temporal chaos and also for the study of other open-flow hydrodynamic systems.

The object of this research-oriented monograph is to summarize and report past and recent developments of the modeling of falling liquid films subjected or not to heat transfer. But because falling films are part of the general class of interfacial flows, we also outline the fundamentals of interfacial fluid mechanics. The conceptual framework, the underlying assumptions and the associated limits of applicability of the different methodologies are systematically given at each step of the

derivations with the aim a ready-to-use text with easy access to mathematical models of different degrees of complexity. Details of the basic numerical methods we used, as well as an introduction to the software package *AUTO-07P* for continuation and bifurcation problems in ordinary differential equations [79], are provided in appendices and tutorials, with the purpose of offering easy access to the falling film area of research. These methods have other uses as well. For instance, the numerical solution of the Orr–Sommerfeld eigenvalue problem we offer is obviously useful not only for the falling film problem, but for numerous fluid flow problems as well, with or without interfaces. Gathering, ordering and giving a detailed overview and comprehensive, critical and pedagogical analysis of past and most up-to-date theoretical and wherever possible experimental advances on film flows have been demanding tasks in view of the numerous and vigorous efforts on the subject. Our sincere hope is that this monograph would be helpful to students and young scientists interested in the field and to scientists both in industry and academia already working with film flows or in general with interfacial fluid mechanics, hydrodynamic stability and nonlinear waves.

Before going any further, the reader will find useful Appendix A, where we render homage to two key scientists, P.L. Kapitza and C.G.M. Marangoni, who made pioneering contributions on falling liquid films and surface-tension-gradient phenomena. Their works have in turn inspired many applied mathematicians, physicists and engineers on falling liquid films and surface-tension-gradient phenomena, including ourselves.

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