Specialization of Quadratic and Symmetric Bilinear Forms

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Manfred Knebusch

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Translated by Thomas Unger

Springer

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Dedicated to the memory of my teachers

Emil Artin 1898–1962

Hel Braun 1914–1986

Ernst Witt 1911–1991

Preface

A Mathematician Said Who
Can Quote Me a Theorem that's True?
For the ones that I Know
Are Simply not So,
When the Characteristic is Two!

This pretty limerick first came to my ears in May 1998 during a talk by T.Y. Lam on field invariants from the theory of quadratic forms.[1] It is—poetic exaggeration allowed—a suitable motto for this monograph.

What is it about? At the beginning of the seventies I drew up a specialization theory of quadratic and symmetric bilinear forms over fields [32]. Let $\lambda : K \to L \cup \infty$ be a place. Then one can assign a form $\lambda_*(\varphi)$ to a form φ over K in a meaningful way if φ has "good reduction" with respect to λ (see §1.1). The basic idea is to simply apply the place λ to the coefficients of φ, which must therefore be in the valuation ring of λ.

The specialization theory of that time was satisfactory as long as the field L, and therefore also K, had characteristic $\neq 2$. It served me in the first place as the foundation for a theory of generic splitting of quadratic forms [33], [34]. After a very modest beginning, this theory is now in full bloom. It became important for the understanding of quadratic forms over fields, as can be seen from the book [26] of Izhboldin–Kahn–Karpenko–Vishik for instance. One should note that there exists a theory of (partial) generic splitting of central simple algebras and reductive algebraic groups, parallel to the theory of generic splitting of quadratic forms (see [29] and the literature cited there).

In this book I would like to present a specialization theory of quadratic and symmetric bilinear forms with respect to a place $\lambda : K \to L \cup \infty$, without the assumption that $\operatorname{char} L \neq 2$. This is where complications arise. We have to make a distinction

[1] "Some reflections on quadratic invariants of fields", 3 May 1998 in Notre Dame (Indiana) on the occasion of O.T. O'Meara's 70th birthday.

between bilinear and quadratic forms and study them both over fields and valuation rings. From the viewpoint of reductive algebraic groups, the so-called regular quadratic forms (see below) are the natural objects. But, even if we are interested only in such forms, we have to know a bit about specialization of nondegenerate symmetric bilinear forms, since they occur as "multipliers" of quadratic forms: if φ is such a bilinear form and ψ is a regular quadratic form, then we can form a tensor product $\varphi \otimes \psi$, see §1.5. This is a quadratic form, which is again regular when ψ has even dimension ($\dim \psi$ = number of variables occurring in ψ). However—and here already we run into trouble—when $\dim \psi$ is odd, $\varphi \otimes \psi$ is not necessarily regular.

Even if we only want to understand quadratic forms over a field K of characteristic zero, it might be necessary to look at specializations with respect to places from K to fields of characteristic 2, especially in arithmetic investigations. When K itself has characteristic 2, an often more complicated situation may occur, for which we are not prepared by the available literature. Certainly fields of characteristic 2 were already allowed in my work on specializations in 1973 [32], but from today's point of view satisfactory results were only obtained for symmetric bilinear forms. For quadratic forms there are gaping holes. We have to study quadratic forms over a valuation ring in which 2 is not a unit. Even the beautiful and extensive book of Ricardo Baeza [6] doesn't give us enough for the theory of specializations, although Baeza even allows semilocal rings instead of valuation rings. He studies only quadratic forms whose associated bilinear forms are nondegenerate. This forces those forms to have even dimension.

Let me now discuss the contents of this book. After an introduction to the problem in §1.1, which can be understood without any previous knowledge of quadratic and bilinear forms, the specialization theory of symmetric bilinear forms is presented in §1.2–§1.3. There are good, generally accessible sources available for the foundations of the algebraic theory of symmetric bilinear forms. Therefore many results are presented without a proof, but with a reference to the literature instead. As an important application, the outlines of the theory of generic splitting in characteristic $\neq 2$ are sketched in §1.4, nearly without proofs.

From §1.5 onwards we address the theory of quadratic forms. In characteristic 2 fewer results can be found in the literature for such forms than for bilinear forms, even at the basic level. Therefore we present most of the proofs. We also concern ourselves with the so-called "weak specialization" (see §1.1) and get into areas which may seem strange even to specialists in the theory of quadratic forms. In particular we have to require a quadratic form over K to be "obedient" in order to weakly specialize it with respect to a place $\lambda : K \to L \cup \infty$ (see §1.7). I have never encountered such a thing anywhere in the literature.

At the end of Chapter 1 we reach a level in the specialization theory of quadratic forms that facilitates a generic splitting theory, useful for many applications. In the first two sections (§2.1, §2.2) of Chapter 2 we produce such a generic splitting theory in two versions, both of which deserve interest in their own right.

We call a quadratic form φ over a field k *nondegenerate* when its quasilinear part (cf. Arf [3]), which we denote by $QL(\varphi)$, is anisotropic. We further call—deviating from Arf [3]—φ *regular* when $QL(\varphi)$ is at most one-dimensional and *strictly regular* when $QL(\varphi) = 0$ (cf. §1.6, Definition 1.59). When k has characteristic $\neq 2$, every nondegenerate form is strictly regular, but in characteristic 2 the quasilinear part causes complications. For in this case φ can become degenerate under a field extension $L \supset k$. Only in the regular case is this impossible.

In §2.1 we study the splitting behaviour of a regular quadratic form φ over k under field extensions, while in §2.2 any nondegenerate form φ, but only separable extensions of k are allowed. The theory of §2.1 incorporates the theory of §1.4, so the missing proofs of §1.4 are subsequently filled in.

Until the end of §2.2 our specialization theory is based on an obvious "canonical" concept of *good reduction* of a form φ over a field K (quadratic or symmetric bilinear) to a valuation ring $\mathfrak{o}$ of K, similar to what is known under this name in other areas of mathematics (e.g. abelian varieties). There is nothing wrong with this theory; however, for many applications it is too limited.

This is particularly clear when studying specializations with respect to a place $\lambda : K \to L \cup \infty$ with $\operatorname{char} K = 0$, $\operatorname{char} L = 2$. If φ is a nondegenerate quadratic form over K with good reduction with respect to λ, then the specialization $\lambda_*(\varphi)$ is automatically strictly regular. However, we would like to have a more general specialization concept, in which forms with quasilinear part $\neq 0$ can arise over L. Conversely, if the place λ is surjective, i.e. $\lambda(K) = L \cup \infty$, we would like to "lift" *every* nondegenerate quadratic form ψ over L with respect to λ to a form φ over K, i.e. to find a form φ over K which specializes to ψ with respect to λ. Then we could use the theory of forms over K to make statements about ψ.

We present such a general specialization theory in §2.3. It is based on the concept of "*fair reduction*", which is less orthodox than good reduction, but which nevertheless possesses quite satisfying properties.

Next, in §2.4, we present a theory of generic splitting, which unites the theories of §1.4, §2.1 and §2.2 under one roof and which incorporates fair reduction. This theory is deepened in §2.5 and §2.6 through the study of generic splitting towers, and thus we reach the end of Chapter 2.

Chapter 3 (§3.1–§3.13) is a long chapter in which we present a panorama of results about quadratic forms over fields for which specialization and generic splitting of forms play an important role. This only scratches the surface of applications of the specialization theory of Chapters 1 and 2. Certainly many more results can be unearthed.

We return to the foundations of specialization theory in the final short Chapter 4 (§4.1–§4.5). Quadratic and bilinear forms over a field can be specialized with respect to a more general "quadratic place" $\Lambda : K \to L \cup \infty$ (defined in §4.1) instead of a usual place $\lambda : K \to L \cup \infty$. This represents a considerable broadening of the specialization theory of Chapters 1 and 2. Of course we require again "obedience" from a quadratic form q over K in order for its specialization $\Lambda_*(q)$ to reasonably exist. It then turns out that the generic splitting behaviour of $\Lambda_*(q)$ is governed by the splitting behaviour of q and Λ, in so far as good or fair reduction is present in a weak sense, as elucidated for ordinary places in Chapter 2.

Why are quadratic places of interest, compared with ordinary places? To answer this question we observe the following. If a form q over K has bad reduction with respect to a place $\lambda : K \to L \cup \infty$, it often happens that λ can be "enlarged" to a quadratic place $\Lambda : K \to L \cup \infty$ such that q has good or fair reduction with respect to Λ in a weak sense, and the splitting properties of q are handed down to $\Lambda_*(q)$ while there is no form $\lambda_*(q)$ available for which this would be the case. The details of such a notion of reduction are much more tricky compared with what happens in Chapters 1 and 2. The central term which renders possible a unified theory of generic splitting of quadratic forms is called "stably conservative reduction", see §4.4.

One must get used to the fact that for bilinear forms there is in general no Witt cancellation rule, in contrast to quadratic forms. Nevertheless the specialization theory is in many respects easier for bilinear forms than for quadratic forms.

On the other hand we do not have any theory of generic splitting for symmetric bilinear forms over fields of characteristic 2. Such a theory might not even be possible in a meaningful way. This may well be connected to the fact that the automorphism groups of such forms can be very far from being reductive groups (which may also account for the absence of a good cancellation rule).

This book is intended for audiences with different interests. For a mathematician with perhaps only a little knowledge of quadratic or symmetric bilinear forms, who just wants to get an impression of specialization theory, it suffices to read §1.1–§1.4. The theory of generic splitting in characteristic $\neq 2$ will acquaint the reader with an important application area.

From §1.5 onwards the book is intended for scholars working in the algebraic theory of quadratic forms, and also for specialists in the area of algebraic groups. They have always been given something to look at by the theory of quadratic forms.

On reaching §2.2 of the book, readers can lean back in their chair and take a well-deserved break. They will have learned about the specialization theory, which is based on the concept of good reduction, and will have gained a certain perspective on specific phenomena in characteristic 2. Furthermore, they will have been introduced to the foundations of generic splitting and so will have seen the specialization theory in action. Admittedly, readers will not yet have seen independent applications of the weak specialization theory (§1.3, §1.7), for this theory has only appeared up to then as an auxiliary one.

The remaining sections §2.3–§2.6 of Chapter 2 develop the specialization theory sufficiently far to allow an understanding of the classical algebraic theory of quadratic forms (as presented in the books of Lam [43], [44] and Scharlau [55]) without the usual restriction that the characteristic should be different from 2. Precisely this happens in Chapter 3 where readers will also obtain sufficient illustrations to enable them to relieve other classical theorems from the characteristic $\neq 2$ restriction, although this is often a nontrivial task.

The final Chapter 4 is ultimately intended for mathematicians who want to embark on a more daring expedition in the realm of quadratic forms over fields. It cannot be mere coincidence that the specialization theory for quadratic places works just as well as the specialization theory for ordinary places. It is therefore a safe prediction that quadratic places will turn out to be generally useful and important in a future theory of quadratic forms over fields.

Regensburg, June 2007 *Manfred Knebusch*

Postscript (October 2009)

I had the very good luck to find a translator of the German text into English, who, besides having two languages, could also understand the mathematical content of this book in depth. I met Professor Thomas Unger within the framework of the European network "Linear Algebraic Groups, Algebraic K-theory, and Related Topics", and most of the translation and our collaboration has been done under the auspices of this network, which we acknowledge gratefully.

I further owe deep thanks to my former secretary Rosi Bonn, who typed the whole German text in various versions and large parts of the English text.

The German text, *Spezialisierung von quadratischen und symmetrischen bilinearen Formen*, can be found on my homepage.[2]

[2] http://www-nw.uni-regensburg.de/∼.knm22087.mathematik.uni-regensburg.de

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