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Probability Theory

A Comprehensive Course

 Springer

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Preface

This book is based on two four-hour courses on advanced probability theory that I have held in recent years at the universities of Cologne and Mainz. It is implicitly assumed that the reader has a certain familiarity with the basic concepts of probability theory, although the formal framework will be fully developed in this book.

The aim of this book is to present the central objects and concepts of probability theory: random variables, independence, laws of large numbers and central limit theorems, martingales, exchangeability and infinite divisibility, Markov chains and Markov processes, as well as their connection with discrete potential theory, coupling, ergodic theory, Brownian motion and the Itô integral (including stochastic differential equations), the Poisson point process, percolation and the theory of large deviations.

Measure theory and integration are necessary prerequisites for a systematic probability theory. We develop it only to the point to which it is needed for our purposes: construction of measures and integrals, the Radon-Nikodym theorem and regular conditional distributions, convergence theorems for functions (Lebesgue) and measures (Prohorov) and construction of measures in product spaces. The chapters on measure theory do not come as a block at the beginning (although they are written such that this would be possible; that is, independent of the probabilistic chapters) but are rather interlaced with probabilistic chapters that are designed to display the power of the abstract concepts in the more intuitive world of probability theory. For example, we study percolation theory at the point where we barely have measures, random variables and independence; not even the integral is needed. As the only exception, the *systematic* construction of independent random variables is deferred to Chapter 14. Although it is rather a matter of taste, I hope that this setup helps to motivate the reader throughout the measure-theoretical chapters.

Those readers with a solid measure-theoretical education can skip in particular the first and fourth chapters and might wish only to look up this or that.

In the first eight chapters, we lay the foundations that will be needed in all subsequent chapters. After that, there are seven more or less independent parts, consisting of Chapters 9–12, 13, 14, 15–16, 17–19, 20 and 23. The chapter on Brownian motion (21) makes reference to Chapters 9–15. Again, after that, the three blocks consisting of Chapters 22, 24 and 25–26 can be read independently.

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I would be grateful for further suggestions, errors etc. to be sent by e-mail to
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Mainz,
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Achim Klenke

Contents

Preface	V
1 Basic Measure Theory	1
1.1 Classes of Sets	1
1.2 Set Functions	12
1.3 The Measure Extension Theorem	18
1.4 Measurable Maps	34
1.5 Random Variables	43
2 Independence	49
2.1 Independence of Events	49
2.2 Independent Random Variables	56
2.3 Kolmogorov's 0-1 Law	63
2.4 Example: Percolation	66
3 Generating Functions	77
3.1 Definition and Examples	77
3.2 Poisson Approximation	80
3.3 Branching Processes	82
4 The Integral	85
4.1 Construction and Simple Properties	85
4.2 Monotone Convergence and Fatou's Lemma	93
4.3 Lebesgue Integral versus Riemann Integral	95

5	Moments and Laws of Large Numbers	101
5.1	Moments	101
5.2	Weak Law of Large Numbers	108
5.3	Strong Law of Large Numbers	111
5.4	Speed of Convergence in the Strong LLN	119
5.5	The Poisson Process	123
6	Convergence Theorems	129
6.1	Almost Sure and Measure Convergence	129
6.2	Uniform Integrability	134
6.3	Exchanging Integral and Differentiation	140
7	L^p-Spaces and the Radon-Nikodym Theorem	143
7.1	Definitions	143
7.2	Inequalities and the Fischer-Riesz Theorem	145
7.3	Hilbert Spaces	151
7.4	Lebesgue's Decomposition Theorem	154
7.5	Supplement: Signed Measures	158
7.6	Supplement: Dual Spaces	165
8	Conditional Expectations	169
8.1	Elementary Conditional Probabilities	169
8.2	Conditional Expectations	173
8.3	Regular Conditional Distribution	179
9	Martingales	189
9.1	Processes, Filtrations, Stopping Times	189
9.2	Martingales	194
9.3	Discrete Stochastic Integral	198
9.4	Discrete Martingale Representation Theorem and the CRR Model ..	200
10	Optional Sampling Theorems	205
10.1	Doob Decomposition and Square Variation	205
10.2	Optional Sampling and Optional Stopping	209

10.3	Uniform Integrability and Optional Sampling	214
11	Martingale Convergence Theorems and Their Applications	217
11.1	Doob's Inequality	217
11.2	Martingale Convergence Theorems	219
11.3	Example: Branching Process	228
12	Backwards Martingales and Exchangeability	231
12.1	Exchangeable Families of Random Variables	231
12.2	Backwards Martingales	236
12.3	De Finetti's Theorem	239
13	Convergence of Measures	245
13.1	A Topology Primer	245
13.2	Weak and Vague Convergence	251
13.3	Prohorov's Theorem	259
13.4	Application: A Fresh Look at de Finetti's Theorem	268
14	Probability Measures on Product Spaces	271
14.1	Product Spaces	272
14.2	Finite Products and Transition Kernels	275
14.3	Kolmogorov's Extension Theorem	283
14.4	Markov Semigroups	288
15	Characteristic Functions and the Central Limit Theorem	293
15.1	Separating Classes of Functions	293
15.2	Characteristic Functions: Examples	300
15.3	Lévy's Continuity Theorem	307
15.4	Characteristic Functions and Moments	312
15.5	The Central Limit Theorem	317
15.6	Multidimensional Central Limit Theorem	324
16	Infinitely Divisible Distributions	327
16.1	Lévy-Khinchin Formula	327

16.2	Stable Distributions	339
17	Markov Chains	345
17.1	Definitions and Construction	345
17.2	Discrete Markov Chains: Examples	352
17.3	Discrete Markov Processes in Continuous Time	356
17.4	Discrete Markov Chains: Recurrence and Transience	361
17.5	Application: Recurrence and Transience of Random Walks	365
17.6	Invariant Distributions	372
18	Convergence of Markov Chains	379
18.1	Periodicity of Markov Chains	379
18.2	Coupling and Convergence Theorem	383
18.3	Markov Chain Monte Carlo Method	390
18.4	Speed of Convergence	398
19	Markov Chains and Electrical Networks	403
19.1	Harmonic Functions	404
19.2	Reversible Markov Chains	407
19.3	Finite Electrical Networks	408
19.4	Recurrence and Transience	414
19.5	Network Reduction	421
19.6	Random Walk in a Random Environment	427
20	Ergodic Theory	431
20.1	Definitions	431
20.2	Ergodic Theorems	435
20.3	Examples	437
20.4	Application: Recurrence of Random Walks	439
20.5	Mixing	442
21	Brownian Motion	447
21.1	Continuous Versions	447
21.2	Construction and Path Properties	454

21.3	Strong Markov Property	459
21.4	Supplement: Feller Processes	462
21.5	Construction via L^2 -Approximation	465
21.6	The Space $C([0, \infty))$	469
21.7	Convergence of Probability Measures on $C([0, \infty))$	471
21.8	Donsker's Theorem	474
21.9	Pathwise Convergence of Branching Processes*	477
21.10	Square Variation and Local Martingales	483
22	Law of the Iterated Logarithm	495
22.1	Iterated Logarithm for the Brownian Motion	495
22.2	Skorohod's Embedding Theorem	498
22.3	Hartman-Wintner Theorem	503
23	Large Deviations	505
23.1	Cramér's Theorem	506
23.2	Large Deviations Principle	510
23.3	Sanov's Theorem	514
23.4	Varadhan's Lemma and Free Energy	519
24	The Poisson Point Process	525
24.1	Random Measures	525
24.2	Properties of the Poisson Point Process	529
24.3	The Poisson-Dirichlet Distribution*	535
25	The Itô Integral	543
25.1	Itô Integral with Respect to Brownian Motion	543
25.2	Itô Integral with Respect to Diffusions	551
25.3	The Itô Formula	554
25.4	Dirichlet Problem and Brownian Motion	562
25.5	Recurrence and Transience of Brownian Motion	564
26	Stochastic Differential Equations	567
26.1	Strong Solutions	567

26.2 Weak Solutions and the Martingale Problem	576
26.3 Weak Uniqueness via Duality	583
References	591
Notation Index	599
Name Index	603
Subject Index	607