Computational Methods for Controller Design
The monograph describes a computationally-based methodology for controller design that can handle typical time and frequency domain specifications and provide a characterization of the limits of performance for a given system. The methodology is based on optimization, where one performance objective is optimized and the other specifications are the constraints. Many practical performance objectives can be represented in terms of convex constraints on the closed loop system response. Such objectives include standard norm constraints, bandwidth constraints, time templates on the closed loop response, as well as stability robustness conditions for unstructured uncertainty. With this, the design problem is turned into an infinite dimensional convex optimization problem. The infinite dimension arises from the fact that the set of feasible closed loop maps (corresponding to a stabilizing controller) is infinite.

A key issue in this book is understanding how do we solve these problems. The solvability of a problem describes when and in what sense the solution can be computed with reasonable complexity.

While finite dimensional convex problems can be efficiently solved, the situation is more complicated with infinite dimensional ones. There is a large class of problems that have an underlying finite dimensional structure. It is interesting to note that in the $\mathcal{H}_\infty$ and $\ell_1$ methodologies, one-block problems belonged to this class. Characterizing such classes is an important component of this research.

When the problem is truly infinite dimensional, or when no finite dimensional structure can be revealed, the solvability of the problem is characterized by the existence of computable approximate solutions, the accuracy of the approximations, and by the information provided by the approximate solutions about the structure of the optimal solution or the optimal controller.

It turns out that duality theory is a fundamental tool in analyzing the solvability of infinite dimensional generalized linear programs, and in providing generic computational methods for such problems.

In this book, we develop a uniform treatment of multi-objective control problems by providing

- a unified way to pose the problems as generalized linear programs and to derive their duals,
- duality theory results that characterize the duality relationship for the generalized linear programs arising from multi-objective control problems,
• a set of tools to analyze the convergence properties of the computational method based on the duality relationship,
• the complete analysis and extension of methods developed for the $\ell_1$ problem, for several important multi-objective problems which makes them readily implementable and usable for design.

In summary, the book provides the reader with a rather complete guide on how pose practical multi-objective control problems in this framework, and how to solve them, i.e., how to derive and analyze readily implementable computational methods to derive exact or approximate solutions.

Organization

The monograph is organized in ten chapters. Chapter 1 contains the introduction to the material in the book, and gives an overview on the area of computational approaches to controller design. Chapter 2 contains most of the necessary notation and the mathematical preliminaries that are needed in the development of the investigation. Chapter 3 contains the control problem setup, the description of the stability constraints and of several typical performance objectives, and shows how these specifications are equivalent to generalized linear constraints. In Chapter 4, the duality theory results for the generalized linear programming problems are derived. These results are applied in the following three chapters to the analysis of several multi-objective problems. Chapter 5 contains the complete treatment for the $\ell_1$ control problem with time-domain constraints on the response to fixed inputs. Two cases are considered: 1) the constraints are imposed only for finite time (finite horizon case). 2) The constraints are imposed for infinite time (infinite horizon case). Chapter 6 contains the solvability analysis of the $\ell_1$ optimal control problem with frequency point magnitude constraints. This problem is a convex optimization with infinite dimensional Linear Matrix Inequality constraints. The main point in this chapter is that the fact that primal and dual problem have the same cost (no of duality gap) in an infinite dimensional problem does not imply that we can compute converging primal and dual finite dimensional approximations. Chapter 7 analyzes the mixed $H_2/\ell_1$ problem. The derivations for the mixed $\ell_1/H_2$ problem are analogous and are omitted. In Chapter 8, a new computational method for $\ell_1$ is presented and its properties are analyzed. The issues of deriving exact or approximate solutions are similar when the problems are posed as dynamic games in state space instead of as convex optimizations on the space of the closed loop maps. Chapter 9 contains a dynamic programming derivation of the optimal (nonlinear) static full state feedback strategy that minimizes the worst-case peak-to-peak gain of the closed loop system. Both finite and infinite horizon problems are considered. Once again, that duality theory provides important extra information about the problem
and allows the derivation of the structure of the optimal strategy and of approximation methods when the optimal strategy cannot be computed exactly. Finally, Chapter 10 presents the conclusions.

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