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Roger Howe • Markus Hunziker
Jeb F. Willenbring
Editors

# Symmetry: Representation Theory and Its Applications

## In Honor of Nolan R. Wallach

Birkhäuser

*Editors*
Roger Howe
Department of Mathematics
Yale University
New Haven, CT, USA

Markus Hunziker
Department of Mathematics
Baylor University
Waco, TX, USA

Jeb F. Willenbring
Department of Mathematical Sciences
University of Wisconsin-Milwaukee
Milwaukee, WI, USA

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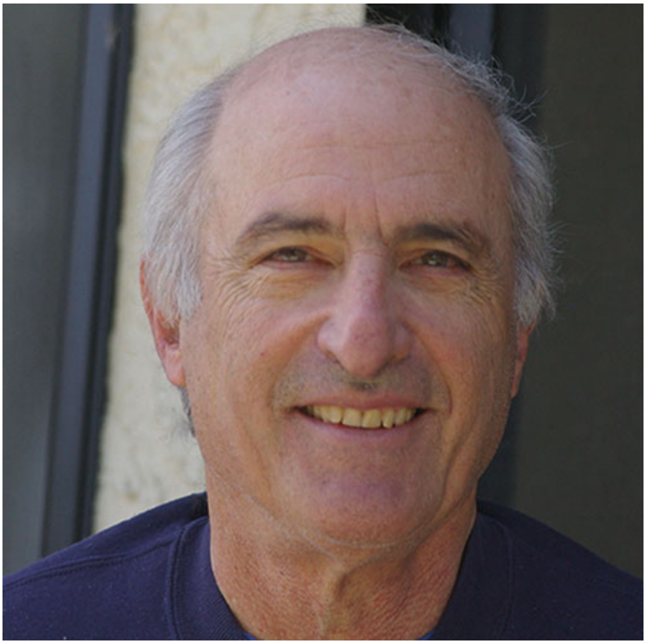
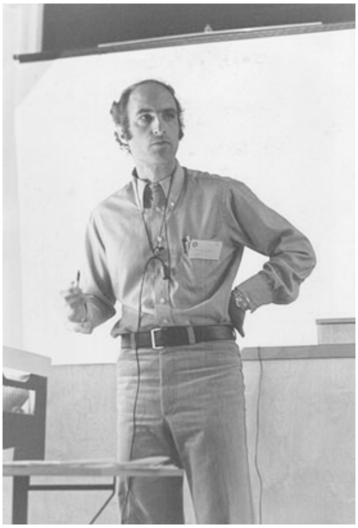


Nolan R. Wallach (2013)

Nolan and Barbara Wallach, Wedding (1965)



Nolan R. Wallach, Helsinki (ICM 1978)

At the UCSD 2011 Conference, surrounding Nolan Wallach (standing in front row) are his students and postdoctoral fellows: Alvany Rocha, Roberto Miatello, Markus Hunziker, Andrew Linshaw, Mandy Cheung, Asif Shakeel, Mark Colarusso, Jon Middleton, Seung Lee, Danny Goldstein, Oded Yacobi, Karin Baur, Gilad Gour, Rino Sanchez, Jeb Willenbring, Sam Evens, Laura Barberis, Allan Keeton, Pierluigi Möseneder Frajria, Jochen Kuttler.

# Preface

Nolan Wallach's contributions to mathematics, made over a period now stretching to nearly five decades, exhibit a breadth of knowledge, research and scholarship matched by few contemporary mathematicians. Although all his research is rooted in his love of geometry and symmetry, one can perhaps usefully distinguish four periods during which this underlying theme was expressed in distinctive ways.

In the first period, roughly 1965–1972, the main motivation was Riemannian geometry (spaces of positive curvature and minimal submanifolds) leading to Do Carmo–Wallach theory, Aloff–Wallach spaces, and Wallach manifolds.

In the second period (1972–1980) the focus of his research moved into the theory of infinite-dimensional representations of semisimple Lie groups and also certain infinite-dimensional groups, with emphasis on unitary representations. He discovered what is now called the Wallach set of unitary positive energy representations. He also began his research into homological methods in representation theory with an eye on applications to the theory of automorphic forms, including collaboration with Armand Borel to produce *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*.

In the third period (1980–1992), Nolan's attention moved to the analytic aspects of representation theory. A notable result was the Casselman–Wallach theorem, which affirms the unity of the algebraic and analytic viewpoints toward irreducible admissible representations of semisimple Lie groups. In harmonic analysis, he gave a proof of the Whittaker Plancherel theorem. He also began his long and fruitful collaboration with Roe Goodman. This period culminated with the publication of *Real Reductive Groups II*, the second volume of his masterful synthesis of the many remarkable developments in representation theory during the 1980s.

In the fourth period, since 1992, Nolan's research shows a renewed interest in algebraic geometry and physics. Here his main emphasis involves applying mathematics related to geometric invariant theory over the real and complex numbers to representation theory, combinatorics and to physics, in particular to quantum information theory. His continuing collaboration with Roe Goodman resulted in publication of the widely influential *Representations and Invariants of the Classical Groups* (which might be thought of as a *Jugendtraum* from his early days as a differential geometer).

Nolan's early training was in differential geometry, and included the study of Lie theory. His thesis advisor was Jun-Ichi Hano, whose mathematical genealogy goes back to Takagi and Hilbert, and whose research interests included homogeneous bounded complex domains, and studies in infinite-dimensional representation theory. Nolan's thesis under Hano dealt with root systems and included a new approach to the classification of real simple Lie algebras.

While Nolan's early papers dealt primarily with differential geometry, Lie theory, and representation theory in particular, is used generously. With M. Do Carmo, he studied minimal immersions of spheres into spheres by making use of spherical harmonics. He studied homogeneous manifolds of positive curvature, including a classification of the even-dimensional ones. Perhaps the most striking contribution from this phase of his work is his construction with S. Aloff of an infinite family of 7-dimensional homogeneous spaces for $SU_3$ with strictly positive curvature.

His training in Lie theory made the transition from geometry to representation theory easy, and in fact, representation theory is already the focus of some of his earliest papers. When he moved to Rutgers after a postdoctoral job at Berkeley, the Institute for Advanced Study was within easy reach, and Nolan visited there frequently. He became one of the most faithful attendees at Harish-Chandra's annual lecture series on representation theory. Nolan also engaged with other faculty at IAS, notably Armand Borel. As mentioned above, they produced a major book on $(\mathfrak{g}, K)$-cohomology of representations and its application to the cohomology of locally symmetric spaces. The first edition of their book appeared as an *A*nnals of Mathematics Study in 1980. A second edition was published as Mathematical Survey and Monograph of the American Mathematical Society.

Nolan worked at Rutgers for twenty years. While there he collaborated with a large number of colleagues both at Rutgers and elsewhere, on a great variety of topics: with many people, including J. Lepowsky, K. Johnson, T.J. Enright, and A. Knapp on representation theory; with S. Greenfield on partial differential equations; with D. DeGeorge and R. Hotta on automorphic forms; with Armand Borel, as described above. He also found time to make discoveries on his own, including his analysis of the "analytic continuation" of the holomorphic discrete series, in which he identified what is now known as the Wallach set—the set of parameters when the analytic continuation of the scalar holomorphic discrete series has a unitary irreducible quotient. This work was in some sense completed in a 1983 paper with T.J. Enright and R. Howe, classifying all holomorphic unitary representations of Hermitian symmetric groups.

When completely integrable systems became the rage in the early 1980s, Nolan quickly assimilated the ideas involved, and wrote a number of papers, with his student A. Rocha-Caridi, on his own, and with his colleague Roe Goodman. This was the beginning of a long and fruitful collaboration with Goodman, resulting in over 10 joint works, including their widely used book on invariant theory of the classical groups. The second edition of this book has been published as a Springer Graduate Text, under the title *Symmetry, Representations, and Invariants*.

The late 1970s and 1980s saw rapid progress in the representation theory of reductive groups, with the ideas of cohomological induction, $D$-modules, and the Kazhdan–Lusztig conjectures, as well as the classifications of irreducible admissible representations by Langlands, by Vogan, and by Beilinson–Bernstein. Nolan made key contributions, including a unification of the analytic and algebraic approaches to representation theory by proving, partly in collaboration with B. Casselman, that all Harish-Chandra modules, representations of a reductive Lie algebra which served as algebraic proxies for infinite-dimensional group representations, could in fact be globalized, in a more or less unique way, to a representation of the associated group. He also contributed a very simple argument that cohomologically induced representations could be unitarized under suitable restrictions on the parameters involved. As noted above, many of these developments are described in his remarkable two-part synthesis of the main foundational results in representation theory, *Real Reductive Groups, I* and *II*, published by Academic Press, which remain the most complete and coherent account of the striking progress due to many people throughout the 1980s; the developments in these volumes also reflect a broad picture of general ideas of representation theory.

In 1989, Nolan, who had been at Rutgers for two decades and had seemed immovable in the face of many attempts to lure him elsewhere, finally succumbed to the attractions of San Diego, and joined the Mathematics Department at UCSD. During his first years at UCSD, he completed *Representations of Reductive Groups, II*, the second volume of his account of the representation theory of reductive Lie groups. In the first volume, he had established the Langlands classification of irreducible admissible representations, the construction of the discrete series, and Harish-Chandra's character theory, including the fundamental regularity theorem, that the characters of irreducible representations, which are distributions, are in fact given by integration against locally $L^1$ functions (for which explicit formulas are in principle available). In the second volume, he builds on these results, plus his joint results with David A. Vogan on intertwining operators, to give a complete account of Harish-Chandra's Plancherel Theorem for reductive Lie groups.

One of the first papers Nolan wrote after completion of that massive project was *Invariant differential operators on a reductive Lie algebra and Weyl group representations* (Journal of the American Mathematical Society, 1993). This paper makes a lovely connection between Harish-Chandra's theory of invariant differential operators on a reductive Lie algebra and the theory of "Springer representations", which associates representations of the Weyl group to nilpotent orbits. It also contains a beautiful new and drastically simpler proof of Harish-Chandra's famous local $L^1$ theorem for invariant eigendistributions on a semisimple Lie algebra. It is a pity that these insights were not available during the writing of *Real Reductive Groups*.

Despite the distance of San Diego from other centers of mathematical research, Nolan has never lacked for visitors. Among the "regulars" have been Benedict Gross, Bertram Kostant, and Hanspeter Kraft. Nolan has coauthored several papers with each of them. With Gross, Nolan constructed a distinguished family of unitary representations for the exceptional groups of real rank $= 4$ by a continuation of "quaternionic discrete series." With Kostant, he developed Gel'fand–Zeitlin theory from the perspective of classical mechanics, and with Kraft, he studied the geometry of finite-dimensional representations.

At UCSD, Nolan also enjoyed the presence of A.M. Garsia with whom he regularly shared ideas and found mutual inspiration. Starting in the 2000s, they began writing papers together, and established several difficult results on quasi-symmetric polynomials and invariant theory.

In the late 1990s, Nolan became interested in quantum information theory, and especially, quantum entanglement. Using his extensive knowledge of invariant theory he was able to make significant contributions to the field. In a 2002 paper with D. Meyer, he defined a simple polynomial measure of multiparticle entanglement which is scalable, i.e., which applies to any number of spin 1/2 particles. A recurring theme in his work on quantum entanglement and invariant theory has been the calculation of explicit Hilbert series. In these computations, he was also able to exercise his considerable expertise in conventional computation to solve problems that not long ago seemed out of reach.

Nolan continues to mentor students, so that now approximately half of his students have degrees from Rutgers, and half from UCSD. We are delighted to recognize and celebrate Nolan's inspiring mathematical journey, which is still very much in progress.

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New Haven, CT, USA — Roger Howe
Waco, TX, USA — Markus Hunziker
Milwaukee, WI, USA — Jeb F. Willenbring
April 2014

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