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With the collaboration of
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Tata Lectures on Theta I

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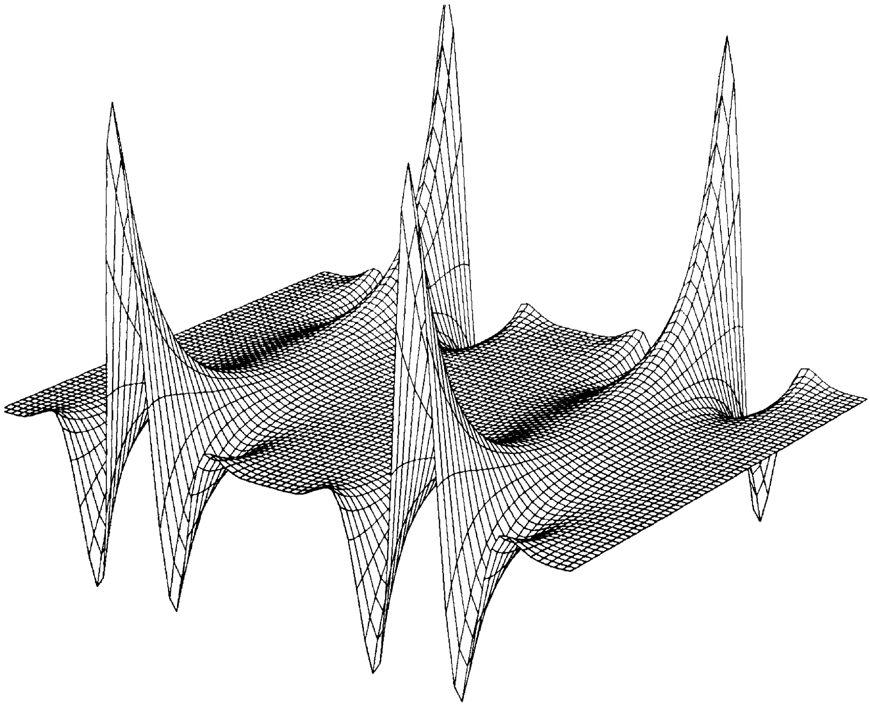

Graph of Re $\vartheta(z, \frac{i}{10})$,

$-0.5 \leq \text{Re } z \leq 1.5$

$-0.3 \leq \text{Im } z \leq 0.3$

TABLE OF CONTENTS

Introduction

This volume contains the first two out of four chapters which are intended to survey a large part of the theory of theta functions. These notes grew out of a series of lectures given at the Tata Institute of Fundamental Research in the period October, 1978, to March, 1979, on which notes were taken and excellently written up by C. Musili and M. Nori. I subsequently lectured at greater length on the contents of Chapter III at Harvard in the fall of 1979 and at a Summer School in Montreal in August, 1980, and again notes were very capably put together by E. Previato and M. Stillman, respectively. Both the Tata Institute and the University of Montreal publish lecture note series in which I had promised to place write-ups of my lectures there. However, as the project grew, it became clear that it was better to tie all these results together, rearranging and consolidating the material, and to make them available from one place. I am very grateful to the Tata Institute and the University of Montreal for permission to do this, and to Birkhauser-Boston for publishing the final result.

The first 2 chapters study theta functions strictly from the viewpoint of classical analysis. In particular, in Chapter I, my goal was to explain in the simplest cases why the theta functions attracted attention. I look at Riemann's theta function $\vartheta(z,\tau)$ for $z \in \mathbb{C}$, $\tau \in H$ = upper half plane, also known as ϑ_{00}, and its 3 variants $\vartheta_{01}, \vartheta_{10}, \vartheta_{11}$. We show how these can be used to embed the torus $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\cdot\tau$ in complex projective 3-space, and how the equations for the image curve can be found. We then prove the functional equation for ϑ with respect to $SL(2,\mathbb{Z})$ and show how from this the moduli space of 1-dimensional tori itself can be realized as an algebraic curve. After this, we prove a beautiful identity of Jacobi on the z-derivative of ϑ. The rest of the chapter is devoted to 3 arithmetic applications of theta series: first to some famous combinatorial identities that follow from the product expansion of ϑ; second to Jacobi's formula for the number of representations of a positive integer as the sum of 4 squares; and lastly to the link between ϑ and ζ and a quick introduction to part of Hecke's theory relating modular forms and Dirichlet series.

The second chapter takes up the generalization of the geometric results of Ch. I (but not the arithmetic ones) to theta functions in several variables, i.e., to $\vartheta(\vec{z},\Omega)$ where $\vec{z} \in \mathbb{C}^g$ and $\Omega \in \mathfrak{H}_g$ = Siegel's g×g upper half-space. Again we show how ϑ can be used to embed the g-dimensional tori X_Ω in projective space. We show how, when Ω is the period matrix of a compact Riemann surface C, ϑ is related to the function theory of C. We prove the functional equation for ϑ and Riemann's theta formula, and sketch how the latter leads to explicit equations for X_Ω as an algebraic variety and to equations for certain modular schemes. Finally we show how from $\vartheta(\vec{z},\Omega)$ a large class of modular forms $\vartheta^{P,Q}(\Omega)$ can be constructed via pluri-harmonic polynomials P and quadratic forms Q.

The third chapter will study theta functions when Ω is a period matrix, i.e., Jacobian theta functions, and, in particular, hyperelliptic theta functions. We will prove an important identity of Fay from which most of the known special identities for Jacobian theta functions follow, e.g., the fact that they satisfy the non-linear differential equation known as the K-P equation. We will study at length the special properties of hyperelliptic theta functions, using an elementary model of hyperelliptic Jacobians that goes back, in its essence, to work of Jacobi himself. This leads us to a characterization of hyperelliptic period matrices Ω by the vanishing of some of the functions $\vartheta[{a \atop b}](0,\Omega)$. One of the goals is to understand hyperelliptic theta functions in their own right well enough so as to be able to deduce directly that functions derived from them satisfy the Korteweg-de Vries equation and other "integrable" non-linear differential equations.

The fourth chapter is concerned with the explanation of the group-representation theoretic meaning of theta functions and the algebro-geometric meaning of theta functions. In particular, we show how $\vartheta(\vec{z},\Omega)$ is, up to an elementary factor, a matrix coefficient of the so-called Heisenberg-Weil representation. And we show how the introduction of finite Heisenberg groups allows one to define theta functions for abelian varieties over arbitrary fields.

The third and fourth chapters will use some algebraic geometry, but the chapters in this volume assume only a knowledge of elementary classical analysis. There are several other excellent books on theta functions available and one might well ask — why another? I wished to bring out several aspects of the theory that I felt were nowhere totally clear: one is the theme that with theta functions many theories that are treated abstractly can be made very concrete and explicit, e.g., the projective embeddings, the equations for, and the moduli of complex tori. Another is the way the Heisenberg group runs through the theory as a unifying thread. However, except for the discussion in Ch. I when $g = 1$, we have not taken up the arithmetic aspects of the theory: Siegel's theory of the representation of one quadratic form by another or the Hecke operators for general g. Nor have we discussed any of the many ideas that have come recently from Shimura's idea of "lifting" modular forms. We want therefore to mention the other important books the reader may consult:

a) J. Fay, Theta Functions on Riemann Surfaces, is the best book on Jacobian theta functions (Springer Lecture Notes 352).

b) E. Freitag, Siegelsche Modulfunktionen, develops the general theory of Siegel modular forms (introducing Hecke operators and the ϕ-operator) and the Siegel modular variety much further.

c) J.-I. Igusa, Theta functions, Springer-Verlag, 1972, like our Chapter IV unifies the group-representation theoretic and algebro-geometric viewpoints. The main result is the explicit projective embedding of the Siegel modular variety by theta constants.

d) G. Lion and M. Vergne, The Weil representation, Maslov index and Theta series (this series, No. 6) discuss the algebra of the metaplectic group on the one hand, and the theory of lifting and the Weil representation on the other. (This is the only treatment of lifting that I have been able to understand.)

The theory of theta functions is far from a finished polished topic. Each chapter finishes with a discussion of some of the unsolved problems. I hope that this book will help to attract more interest to some of these fascinating questions.