

# Vector Bundles and Their Applications

# Mathematics and Its Applications

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# Vector Bundles and Their Applications

by

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## PREFACE

In the last few years the use of geometric methods has permeated many more branches of mathematics and the sciences. Briefly its role may be characterized as follows. Whereas methods of mathematical analysis describe phenomena ‘in the small’, geometric methods contribute to giving the picture ‘in the large’. A second no less important property of geometric methods is the convenience of using its language to describe and give qualitative explanations for diverse mathematical phenomena and patterns. From this point of view, the theory of vector bundles together with mathematical analysis on manifolds (global analysis and differential geometry) has provided a major stimulus. Its language turned out to be extremely fruitful: connections on principal vector bundles (in terms of which various field theories are described), transformation groups including the various symmetry groups that arise in connection with physical problems, in asymptotic methods of partial differential equations with small parameter, in elliptic operator theory, in mathematical methods of classical mechanics and in mathematical methods in economics. There are other currently less significant applications in other fields. Over a similar period, university education has changed considerably with the appearance of new courses on differential geometry and topology. New textbooks have been published but ‘geometry and topology’ has not, in our opinion, been well covered from a practical applications point of view. Existing monographs on vector bundles have been mainly of a purely theoretical nature, devoted to the internal geometric and topological problems of the subject. Students from related disciplines have found the texts difficult to use. It therefore seems expedient to have a simpler book containing numerous illustrations and applications to various problems in mathematics and the sciences.

Part of this book is based on material contained in lectures of the author, A.Mishchenko, given to students of the Mathematics Department at Moscow

State University and is a revised version of the Russian edition of 1984. Some of the less important theorems have been omitted and some proofs simplified and clarified. The focus of attention was towards explaining the most important notions and geometric constructions connected with the theory of vector bundles.

Theorems were not always formulated in maximal generality but rather in such a way that the geometric nature of the objects came to the fore. Whenever possible examples were given to illustrate the role of vector bundles. Thus the book contains sections on locally trivial bundles, and on the simplest properties and operations on vector bundles. Further properties of a homotopic nature, including characteristic classes, are also expounded. Considerable attention is devoted to natural geometric constructions and various ways of constructing vector bundles. Basic algebraic notions involved in describing and calculating K-theory are studied and the particularly interesting field of applications to the theory of elliptic pseudodifferential operators is included. The exposition finishes with further applications of vector bundles to topology. Certain aspects which are well covered in other sources have been omitted in order to prevent the book becoming too bulky.