

Universitext

*Editorial Board
(North America)*

S. Axler
F.W. Gehring
K.A. Ribet

Universitext

Editors (North America): S. Axler, F.W. Gehring, and K.A. Ribet

Aksoy/Khamsi: Nonstandard Methods in Fixed Point Theory
Andersson: Topics in Complex Analysis
Aupetit: A Primer on Spectral Theory
Berberian: Fundamentals of Real Analysis
Booss/Bleecker: Topology and Analysis
Borkar: Probability Theory: An Advanced Course
Böttcher/Silbermann: Introduction to Large Truncated Toeplitz Matrices
Carleson/Gamelin: Complex Dynamics
Cecil: Lie Sphere Geometry: With Applications to Submanifolds
Chae: Lebesgue Integration (2nd ed.)
Charlap: Bieberbach Groups and Flat Manifolds
Chern: Complex Manifolds Without Potential Theory
Cohn: A Classical Invitation to Algebraic Numbers and Class Fields
Curtis: Abstract Linear Algebra
Curtis: Matrix Groups
DiBenedetto: Degenerate Parabolic Equations
Dimca: Singularities and Topology of Hypersurfaces
Edwards: A Formal Background to Mathematics I a/b
Edwards: A Formal Background to Mathematics II a/b
Foulds: Graph Theory Applications
Friedman: Algebraic Surfaces and Holomorphic Vector Bundles
Fuhrmann: A Polynomial Approach to Linear Algebra
Gardiner: A First Course in Group Theory
Gårding/Tambour: Algebra for Computer Science
Goldblatt: Orthogonality and Spacetime Geometry
Gustafson/Rao: Numerical Range: The Field of Values of Linear Operators and Matrices
Hahn: Quadratic Algebras, Clifford Algebras, and Arithmetic Witt Groups
Holmgren: A First Course in Discrete Dynamical Systems
Howe/Tan: Non-Abelian Harmonic Analysis: Applications of $SL(2, \mathbb{R})$
Howes: Modern Analysis and Topology
Hsieh/Sibuya: Basic Theory of Ordinary Differential Equations
Humi/Miller: Second Course in Ordinary Differential Equations
Hurwitz/Kritikos: Lectures on Number Theory
Jennings: Modern Geometry with Applications
Jones/Morris/Pearson: Abstract Algebra and Famous Impossibilities
Kannan/Krueger: Advanced Analysis
Kelly/Matthews: The Non-Euclidean Hyperbolic Plane
Kostrikin: Introduction to Algebra
Luecking/Rubel: Complex Analysis: A Functional Analysis Approach
MacLane/Moerdijk: Sheaves in Geometry and Logic
Marcus: Number Fields
McCarthy: Introduction to Arithmetical Functions
Meyer: Essential Mathematics for Applied Fields
Mines/Richman/Ruitenburg: A Course in Constructive Algebra
Moise: Introductory Problems Course in Analysis and Topology
Morris: Introduction to Game Theory
Polster: A Geometrical Picture Book
Porter/Woods: Extensions and Absolutes of Hausdorff Spaces

(continued after index)

Fuzhen Zhang

Matrix Theory

Basic Results and Techniques



Springer

Fuzhen Zhang
Department of Math, Science, and Technology
Nova Southeastern University
Fort Lauderdale, FL 33314
USA

Editorial Board
(North America)

S. Axler
Mathematics Department
San Francisco State University
San Francisco, CA 94132
USA

F.W. Gehring
Mathematics Department
East Hall
University of Michigan
Ann Arbor, MI 48109
USA

K.A. Ribet
Department of Mathematics
University of California at Berkeley
Berkeley, CA 94720-3840
USA

Mathematics Subject Classification (1991): 15-01

Library of Congress Cataloging-in-Publication Data

Zhang, Fuzhen, 1961–

Matrix theory : basic results and techniques / Fuzhen Zhang.

p. cm. — (Universitext)

Includes bibliographical references and index.

ISBN 978-1-4757-5799-6 ISBN 978-1-4757-5797-2 (eBook)

DOI 10.1007/978-1-4757-5797-2

1. Matrices. I. Title

QA188.Z47 1999

512.9'434—dc21

98-51754

Printed on acid-free paper.

© 1999 Springer Science+Business Media New York

Originally published by Springer-Verlag New York, Inc. in 1999

Softcover reprint of the hardcover 1st edition 1999

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher Springer Science+Business Media, LLC.

except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

Production managed by Terry Kornak; manufacturing supervised by Jeffrey Taub.

Camera-ready copy prepared by the author.

9 8 7 6 5 4 3 2 1

ISBN 978-1-4757-5799-6

SPIN 10707272

To my wife Cheng, daughter Sunny, and son Andrew

Preface

It has been my goal to write a concise book that contains fundamental ideas, results, and techniques in linear algebra and (mainly) in matrix theory which are accessible to general readers with an elementary linear algebra background. I hope this book serves the purpose.

Having been studied for more than a century, linear algebra is of central importance to all fields of mathematics. Matrix theory is widely used in a variety of areas including applied math, computer science, economics, engineering, operations research, statistics, and others.

Modern work in matrix theory is not confined to either linear or algebraic techniques. The subject has a great deal of interactions with combinatorics, group theory, graph theory, operator theory, and other mathematical disciplines. Matrix theory is still one of the richest branches of mathematics; some intriguing problems in the field were long standing, such as the Van der Warden conjecture (1926–1980), and some, such as the permanent-dominance conjecture (since 1966), are still open.

This book contains eight chapters covering various topics from similarity and special types of matrices to Schur complements and matrix normality. Each chapter focuses on the results, techniques, and methods that are beautiful, interesting, and representative, followed by carefully selected problems. Many theorems are given different proofs. The material is treated primarily by matrix approaches and reflects the author's tastes.

The book can be used as a text or a supplement for a linear algebra or matrix theory class or seminar. A one-semester course may consist of the first four chapters plus any other chapter(s) or section(s). The only prerequisites are a decent background in elementary linear algebra and calculus (continuity, derivative, and compactness in a few places). The book can also serve as a reference for researchers and instructors.

The author has benefited from numerous books and journals, including *The American Mathematical Monthly*, *Linear Algebra and Its Applications*, *Linear and Multilinear Algebra*, and the International Linear Algebra Society (ILAS) *Bulletin Image*. This book would not exist without the earlier works of a great number of authors (see the References).

I am grateful to the following professors for many valuable suggestions and input and for carefully reading the manuscript so that many errors have been eliminated from the earlier version of the book:

Professor R. B. Bapat (Indian Statistical Institute),
Professor L. Elsner (University of Bielefeld),
Professor R. A. Horn (University of Utah),

Professor T.-G. Lei (National Natural Science Foundation of China),
Professor J.-S. Li (University of Science and Technology of China),
Professor R.-C. Li (University of Kentucky),
Professor Z.-S. Li (Georgia State University),
Professor D. Simon (Nova Southeastern University),
Professor G. P. H. Styan (McGill University),
Professor B.-Y. Wang (Beijing Normal University), and
Professor X.-P. Zhang (Beijing Normal University).

F.-Z. Zhang
Ft. Lauderdale
March 5, 1999
zhang@polaris.nova.edu
<http://www.polaris.nova.edu/~zhang>

Contents

Preface	vii
Frequently Used Notation and Terminology	xi
Frequently Used Theorems	xiii
1 Elementary Linear Algebra Review	1
1.1 Vector Spaces	1
1.2 Matrices	6
1.3 Linear Transformations and Eigenvalues	14
1.4 Inner Product Spaces	22
2 Partitioned Matrices	29
2.1 Elementary Operations of Partitioned Matrices	29
2.2 The Determinant and Inverse of Partitioned Matrices	36
2.3 The Inverse of a Sum	43
2.4 The Rank of Product and Sum	46
2.5 Eigenvalues of AB and BA	51
2.6 The Continuity Argument	56
3 Matrix Polynomials and Canonical Forms	59
3.1 Commuting Matrices	59
3.2 Matrix Decompositions	64
3.3 Annihilating Polynomials of Matrices	70
3.4 Jordan Canonical Forms	74
3.5 The Matrices A^T , \bar{A} , A^* , $A^T A$, $A^* A$, and $\bar{A} A$	83
3.6 Numerical Range	88
4 Special Types of Matrices	93
4.1 Idempotence, Nilpotence, Involution, and Projection	93
4.2 Tridiagonal Matrices	101
4.3 Circulant Matrices	106
4.4 Vandermonde Matrices	111
4.5 Hadamard Matrices	118
4.6 Permutation and Doubly Stochastic Matrices	123

5	Unitary Matrices and Contractions	131
5.1	Properties of Unitary Matrices	131
5.2	Real Orthogonal Matrices	137
5.3	Metric Space and Contractions	142
5.4	Contractions and Unitary Matrices	148
5.5	The Unitary Similarity of Real Matrices	152
5.6	A Trace Inequality of Unitary Matrices	155
6	Positive Semidefinite Matrices	159
6.1	Positive Semidefinite Matrices	159
6.2	A Pair of Positive Semidefinite Matrices	166
6.3	Partitioned Positive Semidefinite Matrices	175
6.4	Schur Complements and Determinantal Inequalities	184
6.5	The Kronecker Product and Hadamard Product	190
6.6	Schur Complements and Hadamard Products	198
6.7	The Cauchy-Schwarz and Kantorovich Inequalities	203
7	Hermitian Matrices	208
7.1	Hermitian Matrices	208
7.2	The Product of Hermitian Matrices	213
7.3	The Min-Max Theorem and Interlacing Theorem	219
7.4	Eigenvalue and Singular Value Inequalities	227
7.5	A Triangle Inequality for the Matrix $(A^*A)^{\frac{1}{2}}$	235
8	Normal Matrices	240
8.1	Equivalent Conditions	240
8.2	Normal Matrices with Zero and One Entries	251
8.3	A Cauchy-Schwarz Type Inequality for Matrix $(A^*A)^{\frac{1}{2}}$	255
8.4	Majorization and Matrix Normality	260
	References	265
	Notation	273
	Index	275

Frequently Used Notation and Terminology

\mathbb{M}_n	$n \times n$ matrices with complex number entries
\mathbb{C}^n	column vectors of n complex components
$\dim V$	dimension of vector space V
(u, v)	inner product of vectors u and v
$\ x\ $	norm or length of vector x , i.e., $\ x\ = (\sum_i x_i ^2)^{\frac{1}{2}}$
I	identity matrix
$A = (a_{ij})$	matrix A with entries a_{ij}
$\text{rank}(A)$	rank of matrix A
$\text{tr } A$	trace of matrix A
$\det A$	determinant of matrix A
A^{-1}	inverse of matrix A
A^T	transpose of matrix A
\overline{A}	conjugate of matrix A
A^*	conjugate transpose of matrix A , i.e., $A^* = \overline{A}^T$
$ A $	determinant for a block matrix A or matrix $(A^*A)^{\frac{1}{2}}$
$[A]$	principal submatrix of matrix A
$\text{Ker}(A)$	kernel or null of A , i.e., $\text{Ker}(A) = \{x : Ax = 0\}$
$\text{Im}(A)$	image space of A , i.e., $\text{Im}(A) = \{Ax\}$
$\lambda_{\max}(A)$	largest eigenvalue of matrix A
$\sigma_{\max}(A)$	largest singular value of matrix A
$A \geq 0$	A is positive semidefinite
$A \geq B$	$A - B$ is positive semidefinite
$\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$	diagonal matrix with $\lambda_1, \lambda_2, \dots, \lambda_n$ on the diagonal
$A \circ B$	Hadamard product of matrices A and B
$A \otimes B$	Kronecker product of matrices A and B

An $n \times n$ matrix A is said to be

upper-triangular	if all entries below the diagonal are zero
diagonalizable	if $P^{-1}AP$ is diagonal for some invertible matrix P
similar to B	if $P^{-1}AP = B$ for some invertible matrix P
unitarily similar to B	if $U^*AU = B$ for some unitary matrix U
unitary	if $AA^* = A^*A = I$
positive semidefinite	if $x^*Ax \geq 0$ for all vectors $x \in \mathbb{C}^n$
Hermitian	if $A = A^*$
normal	if $A^*A = AA^*$

$\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{M}_n$ if $Ax = \lambda x$ for some nonzero $x \in \mathbb{C}^n$.

Frequently Used Theorems

- **Cauchy-Schwarz inequality:** Let V be an inner product space over a number field (\mathbb{R} or \mathbb{C}). Then for all x and y in V

$$|(x, y)|^2 \leq (x, x)(y, y).$$

Equality holds if and only if x and y are linearly dependent.

- **Theorem on the eigenvalues of AB and BA :** Let A and B be $m \times n$ and $n \times m$ complex matrices, respectively. Then AB and BA have the same nonzero eigenvalues, counting multiplicity. Thus

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

- **Schur triangularization theorem:** For any square matrix A , there exists a unitary matrix U such that U^*AU is upper-triangular.
- **Jordan decomposition theorem:** Let A be an n -square complex matrix. Then there exists an $n \times n$ invertible matrix P such that

$$A = P^*(J_1 \oplus J_2 \oplus \cdots \oplus J_k)P,$$

where each J_i , $i = 1, 2, \dots, k$, is a Jordan block.

- **Spectral decomposition theorem:** Let A be an $n \times n$ normal matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then there exists an $n \times n$ unitary matrix U such that

$$A = U^* \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)U.$$

In particular, if A is positive semidefinite, then $\lambda_i \geq 0$; if A is Hermitian, then the λ_i are real; and if A is unitary, then $|\lambda_i| = 1$.

- **Singular value decomposition theorem:** Let A be an $m \times n$ complex matrix with rank r . Then there exist an $m \times m$ unitary matrix U and an $n \times n$ unitary matrix V such that

$$A = UDV,$$

where D is the $m \times n$ matrix with (i, i) -entries the singular values of A , $i = 1, 2, \dots, r$, and other entries 0. If $m = n$, then D is diagonal.