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(continued after index)

Fuzhen Zhang

Matrix Theory

Basic Results and Techniques

Springer

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To my wife Cheng, daughter Sunny, and son Andrew

Preface

It has been my goal to write a concise book that contains fundamental ideas, results, and techniques in linear algebra and (mainly) in matrix theory which are accessible to general readers with an elementary linear algebra background. I hope this book serves the purpose.

Having been studied for more than a century, linear algebra is of central importance to all fields of mathematics. Matrix theory is widely used in a variety of areas including applied math, computer science, economics, engineering, operations research, statistics, and others.

Modern work in matrix theory is not confined to either linear or algebraic techniques. The subject has a great deal of interactions with combinatorics, group theory, graph theory, operator theory, and other mathematical disciplines. Matrix theory is still one of the richest branches of mathematics; some intriguing problems in the field were long standing, such as the Van der Warden conjecture (1926–1980), and some, such as the permanental-dominance conjecture (since 1966), are still open.

This book contains eight chapters covering various topics from similarity and special types of matrices to Schur complements and matrix normality. Each chapter focuses on the results, techniques, and methods that are beautiful, interesting, and representative, followed by carefully selected problems. Many theorems are given different proofs. The material is treated primarily by matrix approaches and reflects the author's tastes.

The book can be used as a text or a supplement for a linear algebra or matrix theory class or seminar. A one-semester course may consist of the first four chapters plus any other chapter(s) or section(s). The only prerequisites are a decent background in elementary linear algebra and calculus (continuity, derivative, and compactness in a few places). The book can also serve as a reference for researchers and instructors.

The author has benefited from numerous books and journals, including *The American Mathematical Monthly, Linear Algebra and Its Applications, Linear and Multilinear Algebra*, and the International Linear Algebra Society (ILAS) Bulletin *Image.* This book would not exist without the earlier works of a great number of authors (see the References).

I am grateful to the following professors for many valuable suggestions and input and for carefully reading the manuscript so that many errors have been eliminated from the earlier version of the book:

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Contents

Frequently Used Notation and Terminology

$\mathbb{M}_n$	$n \times n$ matrices with complex number entries
$\mathbb{C}^n$	column vectors of n complex components
$\dim V$	dimension of vector space V
(u, v)	inner product of vectors u and v
$\|x\|$	norm or length of vector x, i.e., $\|x\| = \left(\sum_i \|x_i\|^2\right)^{\frac{1}{2}}$
I	identity matrix
$A = (a_{ij})$	matrix A with entries a_{ij}
$\operatorname{rank}(A)$	rank of matrix A
$\operatorname{tr} A$	trace of matrix A
$\det A$	determinant of matrix A
A^{-1}	inverse of matrix A
A^T	transpose of matrix A
$\overline{A}$	conjugate of matrix A
A^*	conjugate transpose of matrix A, i.e., $A^* = \overline{A}^T$
$\|A\|$	determinant for a block matrix A or matrix $(A^*A)^{\frac{1}{2}}$
$[A]$	principal submatrix of matrix A
$\operatorname{Ker}(A)$	kernel or null of A, i.e., $\operatorname{Ker}(A) = \{x : Ax = 0\}$
$\operatorname{Im}(A)$	image space of A, i.e., $\operatorname{Im}(A) = \{Ax\}$
$\lambda_{\max}(A)$	largest eigenvalue of matrix A
$\sigma_{\max}(A)$	largest singular value of matrix A
$A \geq 0$	A is positive semidefinite
$A \geq B$	$A - B$ is positive semidefinite
$\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$	diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the diagonal
$A \circ B$	Hadamard product of matrices A and B
$A \otimes B$	Kronecker product of matrices A and B

An $n \times n$ matrix A is said to be

upper-triangular	if all entries below the diagonal are zero
diagonalizable	if $P^{-1}AP$ is diagonal for some invertible matrix P
similar to B	if $P^{-1}AP = B$ for some invertible matrix P
unitarily similar to B	if $U^*AU = B$ for some unitary matrix U
unitary	if $AA^* = A^*A = I$
positive semidefinite	if $x^*Ax \geq 0$ for all vectors $x \in \mathbb{C}^n$
Hermitian	if $A = A^*$
normal	if $A^*A = AA^*$

$\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{M}_n$ if $Ax = \lambda x$ for some nonzero $x \in \mathbb{C}^n$.

Frequently Used Theorems

- **Cauchy-Schwarz inequality:** Let V be an inner product space over a number field ($\mathbb{R}$ or $\mathbb{C}$). Then for all x and y in V

$$|(x, y)|^2 \leq (x, x)(y, y).$$

Equality holds if and only if x and y are linearly dependent.

- **Theorem on the eigenvalues of AB and BA:** Let A and B be $m \times n$ and $n \times m$ complex matrices, respectively. Then AB and BA have the same nonzero eigenvalues, counting multiplicity. Thus

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

- **Schur triangularization theorem:** For any square matrix A, there exists a unitary matrix U such that U^*AU is upper-triangular.

- **Jordan decomposition theorem:** Let A be an n-square complex matrix. Then there exists an $n \times n$ invertible matrix P such that

$$A = P^*(J_1 \oplus J_2 \oplus \cdots \oplus J_k)P,$$

where each J_i, $i = 1, 2, \ldots, k$, is a Jordan block.

- **Spectral decomposition theorem:** Let A be an $n \times n$ normal matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then there exists an $n \times n$ unitary matrix U such that

$$A = U^* \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)U.$$

In particular, if A is positive semidefinite, then $\lambda_i \geq 0$; if A is Hermitian, then the λ_i are real; and if A is unitary, then $|\lambda_i| = 1$.

- **Singular value decomposition theorem:** Let A be an $m \times n$ complex matrix with rank r. Then there exist an $m \times m$ unitary matrix U and an $n \times n$ unitary matrix V such that

$$A = UDV,$$

where D is the $m \times n$ matrix with (i, i)-entries the singular values of A, $i = 1, 2, \ldots, r$, and other entries 0. If $m = n$, then D is diagonal.