

Universitext

*Editorial Board
(North America):*

S. Axler
F.W. Gehring
K.A. Ribet

Springer Science+Business Media, LLC

Universitext

Editors (North America): S. Axler, F.W. Gehring, and K.A. Ribet

Aksoy/Khamsi: Nonstandard Methods in Fixed Point Theory
Andersson: Topics in Complex Analysis
Aupetit: A Primer on Spectral Theory
Bachman/Narici/Beckenstein: Fourier and Wavelet Analysis
Bădescu: Algebraic Surfaces
Balakrishnan/Ranganathan: A Textbook of Graph Theory
Balser: Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations
Bapat: Linear Algebra and Linear Models (2nd ed.)
Berberian: Fundamentals of Real Analysis
Boltyanskii/Efremovich: Intuitive Combinatorial Topology. (Shenitzer, trans.)
Booss-Bleeker: Topology and Analysis
Borkar: Probability Theory: An Advanced Course
Böttcher/Silbermann: Introduction to Large Truncated Toeplitz Matrices
Carleson/Gamelin: Complex Dynamics
Cecil: Lie Sphere Geometry: With Applications to Submanifolds
Chae: Lebesgue Integration (2nd ed.)
Charlap: Bieberbach Groups and Flat Manifolds
Chern: Complex Manifolds Without Potential Theory
Cohn: A Classical Invitation to Algebraic Numbers and Class Fields
Curtis: Abstract Linear Algebra
Curtis: Matrix Groups
Debarre: Higher-Dimensional Algebraic Geometry
DiBenedetto: Degenerate Parabolic Equations
Dimca: Singularities and Topology of Hypersurfaces
Edwards: A Formal Background to Mathematics I a/b
Edwards: A Formal Background to Mathematics II a/b
Farenick: Algebras of Linear Transformations
Foulds: Graph Theory Applications
Friedman: Algebraic Surfaces and Holomorphic Vector Bundles
Fuhrmann: A Polynomial Approach to Linear Algebra
Gardiner: A First Course in Group Theory
Gårding/Tambour: Algebra for Computer Science
Goldblatt: Orthogonality and Spacetime Geometry
Gustafson/Rao: Numerical Range: The Field of Values of Linear Operators and Matrices
Hahn: Quadratic Algebras, Clifford Algebras, and Arithmetic Witt Groups
Heinonen: Lectures on Analysis on Metric Spaces
Holmgren: A First Course in Discrete Dynamical Systems
Howe/Tan: Non-Abelian Harmonic Analysis: Applications of $SL(2, \mathbb{R})$
Howes: Modern Analysis and Topology
Hsieh/Sibuya: Basic Theory of Ordinary Differential Equations
Humi/Miller: Second Course in Ordinary Differential Equations
Hurwitz/Kritikos: Lectures on Number Theory
Jennings: Modern Geometry with Applications
Jones/Morris/Pearson: Abstract Algebra and Famous Impossibilities
Kannan/Krueger: Advanced Analysis

(continued after index)

Arlan Ramsay Robert D. Richtmyer

Introduction to Hyperbolic Geometry

With 59 Figures



Springer

Arlan Ramsay
Department of Mathematics
University of Colorado
Boulder, CO 80303
USA

Robert D. Richtmyer
Department of Mathematics
University of Colorado
Boulder, CO 80303
USA

Editorial Board
(North America):

S. Axler
Mathematics Department
San Francisco State University
San Francisco, CA 94132
USA

F.W. Gehring
Mathematics Department
East Hall
University of Michigan
Ann Arbor, MI 48109-1109
USA

K.A. Ribet
Department of Mathematics
University of California at Berkeley
Berkeley, CA 94720-3840
USA

Mathematics of Subject Classifications (1991): 51-01, 51M09

Library of Congress Cataloging-in-Publication Data
Ramsay, Arlan.

Introduction to hyperbolic geometry/Arlan Ramsay, Robert D.
Richtmyer.

p. cm. – (Universitext)

Includes bibliographical references and index.

ISBN 978-0-387-94339-8 ISBN 978-1-4757-5585-5 (eBook)

DOI 10.1007/978-1-4757-5585-5

1. Geometry, Hyperbolic. I. Richtmyer, Robert D. II. Title.

QA685.R18 1994

516.9–dc20

94-25789

Printed on acid-free paper.

© 1995 Springer Science+Business Media New York

Originally published by Springer-Verlag New York, Inc. in 1995

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher Springer Science+Business Media, LLC, except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

Production managed by Natalie Johnson; manufacturing supervised by Gail Simon.

Photocomposed using the authors' TeX files.

9 8 7 6 5 4 3 2

ISBN 978-0-387-94339-8

SPIN 10851291

Preface

This book is an introduction to hyperbolic and differential geometry that provides material in the early chapters that can serve as a textbook for a standard upper division course on hyperbolic geometry. For that material, the students need to be familiar with calculus and linear algebra and willing to accept one advanced theorem from analysis without proof. The book goes well beyond the standard course in later chapters, and there is enough material for an honors course, or for supplementary reading. Indeed, parts of the book have been used for both kinds of courses.

Even some of what is in the early chapters would surely not be necessary for a standard course. For example, detailed proofs are given of the Jordan Curve Theorem for Polygons and of the decomposability of polygons into triangles. These proofs are included for the sake of completeness, but the results themselves are so believable that most students should skip the proofs on a first reading.

The axioms used are modern in character and more “user friendly” than the traditional ones. The familiar real number system is used as an ingredient rather than appearing as a result of the axioms. However, it should not be thought that the geometric treatment is in terms of models: this is an axiomatic approach that is just more convenient than the traditional ones.

The book is appropriate as part of a special curriculum in undergraduate mathematics for exceptional students, designed to prepare them in three years for graduate-level mathematics. Our experience indicates that such students are able to learn mathematics more rapidly and in more depth than average undergraduate mathematics majors.

A principle that has guided the development is that of *mathematics in parallel*. Students can better see the interconnections if they hear about different subjects more-or-less simultaneously. For example, differential equations ought to be introduced right along with calculus; that makes calculus more interesting and more relevant. As another example, the beginnings of group theory should be included with linear algebra, because the permutation and rotation groups are already present, in effect, and it takes only a little further explanation (which the good students like) to lay the

foundations of group theory. As a third example, some complex analysis is useful in calculus (for example power series in a complex variable); even Newton and Leibnitz used complex quantities (with trepidation).

The import of that idea for the present book is in the following:

(1) We have felt free to use methods of analysis, especially calculus and differential equations. In particular, we have used methods from differential geometry and have supplied two chapters on the relevant parts of that subject.

(2) We have assumed a knowledge of complex numbers and the complex plane, also of matrices and groups, up through homomorphism and normal subgroups.

(3) We have included a brief discussion of the connection with the Lorentz transformations of special relativity.

(4) We have taken the attitude that although students ought to know about the synthetic methods of geometric proof (they ought to have learned about it in high school geometry), and although they ought to know that there were gaps in Euclid which were bridged in the late 19th century, especially by Hilbert, and although they ought to know what a proof is and how to construct one, in fact the bright students don't need a whole semester of synthetic proving to achieve that. We have therefore adopted a set of axioms considerably less primitive than the axioms of Euclid or Hilbert. We feel in particular that in modern mathematics it is not necessary to derive the properties of the real number system from the axioms of geometry. (We have discussed that system independently in an appendix.) The purpose of our axioms is to tell the students what hyperbolic geometry *is* in terms that are familiar to them. However, the concepts that the axioms deal with should be intuitively geometrical. Geometry is not just an exercise in abstract logic, but the concepts deal with things that one can visualize to some extent.

(5) We have attempted to be more forthright about the role of models of the hyperbolic plane than some books are. Models were introduced originally for the purely logical purpose of proving that the axioms are self-consistent (assuming that the Euclidean axioms are self-consistent). One defines "points", "lines", "lengths" and "angles" in terms of certain (often rather strange looking) things in Euclidean geometry, and proves that the hyperbolic axioms are formally satisfied by the things in quotation marks. However, until one has proved that the axiom system is categorical, one cannot exclude the possibility that a given model may have additional properties that do *not* follow from the axioms (they merely don't contradict the axioms), while another model may have properties that contradict those of the first model, just as different groups have different properties even though every group satisfies all the axioms of group theory.

It is therefore not permitted to derive general theorems or formulas from a particular model rather than from the axioms (and the good students wouldn't let us get away with it, because they insist on understanding the

logical connections in the subject). Our procedure is to derive the formulas of the hyperbolic plane from the axioms, without use of a model. To do that, we have first to establish the locally Euclidean nature of the hyperbolic plane: certain Euclidean laws are approximately satisfied by small figures, with a relative error that tends to zero as the size of the figures tends to zero. It seems to us that some books fail to make clear that the property of hyperbolic geometry follows from the hyperbolic axioms, not just from an accidental property of some model.

(6) After the formulas have been derived, we discuss models, and we prove that the axioms are categorical, i.e., that all models are isomorphic; the proof uses the formulas, so that categoricalness could not have been established (at least by this method) until the formulas are available. After categoricalness has been established, we can then derive further general results by use of any model.

(7) Since some of the very bright students have heard of Gödel's incompleteness theorem, we feel obliged to make some very brief remarks on the *apparent* conflict between that theorem and categoricalness, which says that the axioms of hyperbolic geometry are in a sense complete, for the purpose of describing the geometry.

(8) We have corrected a prevalent misunderstanding about astronomical parallax.

(9) We have tried to give an overview, not an encyclopedic sort of treatise. For example, most of the book deals with the two-dimensional cases, i.e., with the hyperbolic plane and the differential geometry of surfaces. Our discussion of the cases of more dimensions in Chapter 9 is admittedly quite sketchy, for we believe that most of the important concepts are found in the two-dimensional cases; once these are understood, the generalization contains no new difficulties.

The authors are grateful to our colleague Bill Reinhardt for much patience in discussing the relationship of Gödel's Incompleteness Theorem to the fact that the hyperbolic plane is unique up to isomorphism. Likewise, we thank our colleague Walter Taylor and his class in the fall term of 1993 for many suggestions for improving the ideas or the exposition. Elizabeth Stimmel was stellar in her performance turning the original typescript into \TeX and her patience through numerous revisions. It is impossible to thank her enough. Finally, we thank Bruce Ramsay for invaluable assistance; he developed the software for drawing the figures and used it to produce them. He also gave crucial help in getting the figures into appropriate positions in the text in spite of the fiendish side effects of certain necessary \TeX macros. He contributed much more than anyone can expect from a friend or a son.

Robert Richtmyer
Arlan Ramsay
Boulder, Colorado
August, 1994

Contents

Preface	v
Introduction	1
A Bit of History	1
What Are Axioms For?	2
What Are Models For? (Consistency and Categoricalness)	3
What Is Geometry? (The Problem of Visualization)	5
The Role of Analysis in Geometry (Differential Geometry)	5
Outline of the Following Chapters	7
Chapter 1	
Axioms for Plane Geometry	9
1.1 The Axioms, Definitions and Remarks	9
1.2 Comments on the Unit of Length	15
1.3 Comments on Spherical Geometry	17
<i>Appendix: The Real Number System</i>	18
A.1 Axioms of an Ordered Field	18
A.2 The Rational Subfield	20
A.3 The Completeness Axiom	22
A.4 Categoricalness of the Axioms	23
A.5 Some Models of \mathbb{R}	25
A.6 Categoricalness and the Gödel Incompleteness Theorem	28
Chapter 2	
Some Neutral Theorems of Plane Geometry	30
2.1 Neutral Theorems	30
2.2 Alternate Interior Angles Theorem	31
2.3 Existence of Perpendiculars; Properties of Certain Functions	32
2.4 The Exterior Angle Theorem and Its Consequences	38
2.5 Congruence Criteria for Triangles	41
2.6 Intersections of Lines and Circles	45

x	Contents	
2.7	The Angle Sum of a Triangle	48
2.8	Quadrilaterals	49
2.9	Polygons	52
2.10	Isometries; The Isometry Group	59
Chapter 3		
	Qualitative Description of the Hyperbolic Plane	69
3.1	The Angle of Parallelism and Asymptotic Pencils	69
3.2	Angular Defects of Triangles and Other Polygons	76
3.3	Application to the Angle of Parallelism	82
3.4	Polar Coordinates and Ideal Points at Infinity	83
3.5	Ultraparallel Lines	86
3.6	Isometries	89
3.7	Rotation by a Composition of Translations	93
3.8	Analysis of Isometries by Reflections	94
3.9	Horocycles — A Special Coordinate System	97
3.10	Equidistants	103
3.11	Tiling, Lattices, and Triangulations	104
3.12	Area and Angular Defect; Equivalence of Polygons	114
3.13	A Misunderstanding About Astronomical Parallax	118
3.14	Bounds on Angular Defects of Small Triangles and Quadrilaterals	120
3.15	Length of a Circular Arc and Area (Defect) of a Circular Sector	123
3.16	Bounds for $g(r)$ and $f(r)$ for Small r	127
Chapter 4		
	\mathbb{H}^3 and Euclidean Approximations in \mathbb{H}^2	128
4.1	Summary	128
4.2	Axioms for \mathbb{H}^3	129
4.3	Some Neutral Theorems of 3-Space	130
4.4	Spherical Coordinates	136
4.5	Isometries in \mathbb{H}^3	137
4.6	Asymptotic Bundles and Ideal Points at Infinity	139
4.7	The Horosphere; The Coordinates ξ, η, ζ ; Ideal Rotations	142
4.8	The Euclidean Geometry in a Horosphere	144
4.9	Euclidean Fine Structure of the Hyperbolic Plane	144
Chapter 5		
	Differential Geometry of Surfaces	149
5.1	Parametric Representation of a Surface in Three Dimensions	150

5.2	Lengths of Curves; the Line Element; the Metric Tensor	153
5.3	Abstract Geometric Surfaces; Line Element in the Hyperbolic Plane	157
5.4	Geodesics and the Calculus of Variations	159
5.5	Angles	163
5.6	Parallel Transport of Vectors	166
5.7	Approximate Laws for Very Small Right Triangles	170
5.8	Area	170
5.9	The Gaussian Total Curvature of a Surface	172
5.10	Differentiable Surfaces	176
5.11	Vectors and Tensors	180
5.12	Invariance of the Line Element Under Isometries	188
Chapter 6		
	Quantitative Considerations	190
6.1	The Angle of Parallelism and Horocycles	190
6.2	Differential Equations and Formulas for $g(r)$ and $f(r)$	192
6.3	Formulas for Triangles and Equidistants	195
6.4	Equation of a Line in Polar Coordinates	199
6.5	The Ideal Points at Infinity	200
6.6	Formulas for Isometries in Polar Coordinates	200
Chapter 7		
	Consistency and Categoricalness of the Hyperbolic Axioms; The Classical Models	202
7.1	Models	202
7.2	Definition of \mathcal{S} and the Coordinates ξ, η	203
7.3	The Poincaré Half-Plane Coordinates	203
7.4	The Isometries of \mathcal{S} in Terms of the Half-Plane Coordinates	204
7.5	The Poincaré Half-Plane Satisfies the Hyperbolic Axioms	205
7.6	Consistency of the Hyperbolic Axioms	207
7.7	Categoricalness of the Hyperbolic Axioms	207
7.8	Other Models	209
7.9	The Pseudosphere or Tractroid	213
7.10	A Hyperbolic Model of the Euclidean Plane	215
Chapter 8		
	Matrix Representation of the Isometry Group	218
8.1	Fractional Linear Transformations	218
8.2	Points, Lines, and Curves Invariant Under an Isometry	221
8.3	Simplicity of the Hyperbolic Direct Isometry Group	223
8.4	The Group $SL(2, \mathbb{Z})$ and the Corresponding Tiling	228

Chapter 9	
Differential and Hyperbolic Geometry in More Dimensions	232
9.1 Manifolds	232
9.2 The Line Element, Geodesics, Volume	235
9.3 The Line Element in Hyperbolic 3-Space	235
9.4 The Horosphere Again	236
9.5 Models of \mathbb{H}^3	238
9.6 Indefinite Metrics; Minkowski Geometry	240
Chapter 10	
Connections with the Lorentz Group of Special Relativity	242
10.1 Origin of Special Relativity	242
10.2 Lorentz and Poincaré Groups	246
10.3 Isomorphism of the Restricted Lorentz Group in Two Space Variables and Time with the Direct Isometry Group of the Hyperbolic Plane	248
10.4 A Pseudoparadoxical Feature of the Lorentz Group	250
10.5 Generalization to Three Space Variables and Time	251
10.6 Relativistic Velocity Space	252
Chapter 11	
Constructions by Straightedge and Compass in the Hyperbolic Plane	254
11.1 Definitions and Examples; Quadratic-Surd Fields	255
11.2 Normal Sets of Points	261
11.3 Segment Trisection	264
11.4 Construction of the Angle of Parallelism	265
11.5 Squaring the Circle	266
11.6 Constructibility of All Points Associated with Quadratic-Surd Fields	268
11.7 Construction of Regular Polygons	275
11.8 The Horocompass Gives Nothing New	276
11.9 The Finite Straightedge	278
11.10 A Set of Axioms Omitting Completeness	280
Index	283