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Introduction to Hyperbolic Geometry

With 59 Figures

Springer

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Preface

This book is an introduction to hyperbolic and differential geometry that provides material in the early chapters that can serve as a textbook for a standard upper division course on hyperbolic geometry. For that material, the students need to be familiar with calculus and linear algebra and willing to accept one advanced theorem from analysis without proof. The book goes well beyond the standard course in later chapters, and there is enough material for an honors course, or for supplementary reading. Indeed, parts of the book have been used for both kinds of courses.

Even some of what is in the early chapters would surely not be necessary for a standard course. For example, detailed proofs are given of the Jordan Curve Theorem for Polygons and of the decomposability of polygons into triangles, These proofs are included for the sake of completeness, but the results themselves are so believable that most students should skip the proofs on a first reading.

The axioms used are modern in character and more "user friendly" than the traditional ones. The familiar real number system is used as an ingredient rather than appearing as a result of the axioms. However, it should not be thought that the geometric treatment is in terms of models: this is an axiomatic approach that is just more convenient than the traditional ones.

The book is appropriate as part of a special curriculum in undergraduate mathematics for exceptional students, designed to prepare them in three years for graduate-level mathematics. Our experience indicates that such students are able to learn mathematics more rapidly and in more depth than average undergraduate mathematics majors.

A principle that has guided the development is that of *mathematics in parallel.* Students can better see the interconnections if they hear about different subjects more-or-less simultaneously. For example, differential equations ought to be introduced right along with calculus; that makes calculus more interesting and more relevant. As another example, the beginnings of group theory should be included with linear algebra, because the permutation and rotation groups are already present, in effect, and it takes only a little further explanation (which the good students like) to lay the foundations of group theory. As a third example, some complex analysis is useful in calculus (for example power series in a complex variable); even Newton and Leibnitz used complex quantities (with trepidation).

The import of that idea for the present book is in the following:

(1) We have felt free to use methods of analysis, especially calculus and differential equations. In particular, we have used methods from differential geometry and have supplied two chapters on the relevant parts of that subject.

(2) We have assumed a knowledge of complex numbers and the complex plane, also of matrices and groups, up through homomorphism and normal subgroups.

(3) We have included a brief discussion of the connection with the Lorentz transformations of special relativity.

(4) We have taken the attitude that although students ought to know about the synthetic methods of geometric proof (they ought to have learned about it in high school geometry), and although they ought to know that there were gaps in Euclid which were bridged in the late 19th century, especially by Hilbert, and although they ought to know what a proof is and how to construct one, in fact the bright students don't need a whole semester of synthetic proving to achieve that. We have therefore adopted a set of axioms considerably less primitive than the axioms of Euclid or Hilbert. We feel in particular that in modern mathematics it is not necessary to derive the properties of the real number system from the axioms of geometry. (We have discussed that system independently in an appendix.) The purpose of our axioms is to tell the students what hyperbolic geometry *is* in terms that are familiar to them. However, the concepts that the axioms deal with should be intuitively geometrical. Geometry is not just an exercise in abstract logic, but the concepts deal with things that one can visualize to some extent.

(5) We have attempted to be more forthright about the role of models of the hyperbolic plane than some books are. Models were introduced originally for the purely logical purpose of proving that the axioms are self-consistent (assuming that the Euclidean axioms are self-consistent). One defines "points", "lines", "lengths" and "angles" in terms of certain (often rather strange looking) things in Euclidean geometry, and proves that the hyperbolic axioms are formally satisfied by the things in quotation marks. However, until one has proved that the axiom system is categorical, one cannot exclude the possibility that a given model may have additional properties that do *not* follow from the axioms (they merely don't contradict the axioms), while another model may have properties that contradict those of the first model, just as different groups have different properties even though every group satisfies all the axioms of group theory.

It is therefore not permitted to derive general theorems or formulas from a particular model rather than from the axioms (and the good students wouldn't let us get away with it, because they insist on understanding the logical connections in the subject). Our procedure is to derive the formulas of the hyperbolic plane from the axioms, without use of a model. To do that, we have first to establish the locally Euclidean nature of the hyperbolic plane: certain Euclidean laws are approximately satisfied by small figures, with a relative error that tends to zero as the size of the figures tends to zero. It seems to us that some books fail to make clear that the property of hyperbolic geometry follows from the hyperbolic axioms, not just from an accidental property of some model.

(6) After the formulas have been derived, we discuss models, and we prove that the axioms are categorical, i.e., that all models are isomorphic; the proof uses the formulas, so that categoricalness could not have been established (at least by this method) until the formulas are available. After categoricalness has been established, we can then derive further general results by use of any model.

(7) Since some of the very bright students have heard of Gödel's incompleteness theorem, we feel obliged to make some very brief remarks on the *apparent* conflict between that theorem and categoricalness, which says that the axioms of hyperbolic geometry are in a sense complete, for the purpose of describing the geometry.

(8) We have corrected a prevalent misunderstanding about astronomical parallax.

(9) We have tried to give an overview, not an encyclopedic sort of treatise. For example, most of the book deals with the two-dimensional cases, i.e., with the hyperbolic plane and the differential geometry of surfaces. Our discussion of the cases of more dimensions in Chapter 9 is admittedly quite sketchy, for we believe that most of the important concepts are found in the two-dimensional cases; once these are understood, the generalization contains no new difficulties.

The authors are grateful to our colleague Bill Reinhardt for much patience in discussing the relationship of Gödel's Incompleteness Theorem to the fact that the hyperbolic plane is unique up to isomorphism. Likewise, we thank our colleague Walter Taylor and his class in the fall term of 1993 for many suggestions for improving the ideas or the exposition. Elizabeth Stimmel was stellar in her performance turning the original typescript into TeX and her patience through numerous revisions. It is impossible to thank her enough. Finally, we thank Bruce Ramsay for invaluable assistance; he developed the software for drawing the figures and used it to produce them. He also gave crucial help in getting the figures into appropriate positions in the text in spite of the fiendish side effects of certain necessary TeX macros. He contributed much more than anyone can expect from a friend or a son.

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