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Olivier Debarre

Higher-Dimensional Algebraic Geometry

With 16 Illustrations

Springer

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Preface

This book deals with the classification theory of algebraic varieties. Traditionally, "higher-dimensional" refers to the case of dimension at least 3 (or, more accurately, of any dimension), as opposed to the case of surfaces, where everything is completely understood.

The theory is largely still in progress. However, an amazing quantity of knowledge has accumulated in the past twenty years, through the combined efforts of S. Mori, F. Campana, Y. Kawamata, J. Kollár, Y. Miyaoka, M. Reid, V. Shokurov, and many others. We are now at a stage where understanding the latest developments requires mastering a multitude of complex and subtle notions. One sometimes feels one has to join a private club where people talk freely about, say, the discrepancy of weak Kawamata log terminal pairs or the existence of general elephants.

As a result, the field has remained much more confidential than it should have. Strong students or seasoned mathematicians have been known to flinch before the task of learning the terminology of the field, discouraged by the reputation of technicality surrounding it or simply not knowing where to begin.

My purpose in writing these notes was to provide an easily accessible introduction to the subject. I have therefore not tried to be exhaustive, nor to write a reference book. To begin with, such books already exist (I am thinking of Kollár's book [K1] on rational curves). Secondly, the field is moving ahead rapidly. Thirdly, I am certainly not the expert that such an undertaking would require. I have, on the contrary, selected small parts of the theory and tried to give definitions, proofs, and examples with as many details as possible. This means that the experts will not find anything new here, except maybe some marginal new results on Fermat hypersurfaces (see Exercises 2.5) and Fano varieties with high degree (see Section 5.11).

The material covered in this book falls roughly into three groups.

• The first two chapters are devoted to preparatory and more or less standard definitions and results on intersection numbers of Cartier divisors, nef, big or ample Cartier divisors, and parameter spaces for morphisms from a fixed curve to a fixed variety, with various decorations.

• Chapters 3, 4, and 5 cover various aspects of the geometry of smooth projective varieties with many rational curves. At their heart lie Mori's bend-and-break lemmas in their different incarnations. We study varieties that are uniruled (i.e., covered by rational curves) or rationally connected (i.e., such that two general points can be joined by a rational curve). We give a structure theorem for varieties with nef anticanonical divisor, and prove that Fano varieties are rationally connected. The proof of this last result rests on a very general construction of Campana's, which in our case yields *the rational quotient.*

• In Chapters 6 and 7, we take the first steps toward Mori's minimal model program of classification of algebraic varieties by proving the cone and the contraction theorems. Given a variety X, the idea is to construct a nontrivial morphism $X \to Y$ with connected fibers to make X "simpler." Such a morphism is characterized by the set of numerical equivalence classes of the curves that it contracts, and the problem is to describe which subsets of classes in the cone $\mathrm{NE}(X)$ generated by classes of curves on X can be obtained in this way. This is exactly what the contraction theorem does. The cone theorem gives a geometric description of (the closure of) the cone $\mathrm{NE}(X)$. For smooth varieties, it is obtained as an application of Mori's bend-and-break techniques and undoubtedly represents the present culmination of the whole theory. Unfortunately, the contraction theorem is unattainable with these ideas. Also, it quickly becomes clear that we must allow some kind of singular varieties in the picture (the contraction of a smooth variety is often singular) and Mori's techniques break down in this case. We undertake a completely different approach (initiated by Kawamata) in the last chapter, which allows us to prove the cone and contraction theorems for varieties with canonical singularities (albeit in characteristic zero only). Alas, gone is the geometry, this is a world of cohomological calculations based on the whole machinery of vanishing theorems! This is by far the most difficult part of the book, but also a necessary evil justified by the scope of this new approach.

The "classical" way to state these vanishing theorems is complicated and makes them look rather unnatural. They look much better when they are rephrased within the so-called "logarithmic" framework, so I take this opportunity to briefly introduce the corresponding language. To include a complete treatment of singularities of pairs would, however, have taken me much too far and there already exist very thorough texts on this subtle question (such as [K7]), but I think the reader should know about the most simple-minded aspects of the theory. Similarly, most theorems concerning the minimal model program could also have been stated in the relative case. The proofs are not much more difficult, but I am not sure this degree of generality provides the best first contact with the theory, so I only mention without proofs some of the relevant results.

I have tried to keep the number of concepts that are used but not proved to a minimum (the construction of the space of morphisms and the Kawamata–Viehweg vanishing theorem are two examples). There is no doubt that some of the material presented here is not easy, but I have done my best to make it accessible. Except for some references to EGA in Chapter 5, my preferred source of reference has been Hartshorne's classic book [H1]. Like most of my predecessors, I claim—genuinely—that a reader familiar with [H1] will go through this book without too much trouble (but with work and patience). Behind the technicalities and the sometimes obscure terminology, I hope she (or he) will discover truly remarkable mathematics well worth her (or his) time.

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