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# Springer Series in Statistics

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(continued after index)

James O. Berger

# Statistical Decision Theory and Bayesian Analysis

Second Edition

With 23 Illustrations



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*To Ann, Jill, and Julie*

# Preface

Statistical decision theory and Bayesian analysis are related at a number of levels. First, they are both needed to solve real decision problems, each embodying a description of one of the key elements of a decision problem. At a deeper level, Bayesian analysis and decision theory provide unified outlooks towards statistics; they give a foundational framework for thinking about statistics and for evaluating proposed statistical methods.

The relationships (both conceptual and mathematical) between Bayesian analysis and statistical decision theory are so strong that it is somewhat unnatural to learn one without the other. Nevertheless, major portions of each have developed separately. On the Bayesian side, there is an extensively developed Bayesian theory of statistical inference (both subjective and objective versions). This theory recognizes the importance of viewing statistical analysis conditionally (i.e., treating observed data as known rather than unknown), even when no loss function is to be incorporated into the analysis. There is also a well-developed (frequentist) decision theory, which avoids formal utilization of prior distributions and seeks to provide a foundation for frequentist statistical theory. Although the central thread of the book will be Bayesian decision theory, both Bayesian inference and non-Bayesian decision theory will be extensively discussed. Indeed, the book is written so as to allow, say, the teaching of a course on either subject separately.

Bayesian analysis and, especially, decision theory also have split personalities with regard to their practical orientation. Both can be discussed at a very practical level, and yet they also contain some of the most difficult and elegant theoretical developments in statistics. The book contains a fair amount of material of each type. There is extensive discussion on how to actually do Bayesian decision theory and Bayesian inference, including how

to construct prior distributions and loss functions, as well as how to utilize them. At the other extreme, introductions are given to some of the beautiful theoretical developments in these areas.

The statistical level of the book is formally rather low, in that previous knowledge of Bayesian analysis, decision theory, or advanced statistics is unnecessary. The book will probably be rough going, however, for those without previous exposure to a moderately serious statistics course. For instance, previous exposure to such concepts as sufficiency is desirable. It should also be mentioned that parts of the book are philosophically very challenging; the extreme disagreements that exist among statisticians, concerning the correct approach to statistics, suggest that these fundamental issues are conceptually difficult. Periodic rereading of such material (e.g., Sections 1.6, 4.1, and 4.12), as one proceeds through the book, is recommended.

The mathematical level of the book is, for the most part, at an easy advanced calculus level. Some knowledge of probability is required; at least, say, a knowledge of expectations and conditional probability. From time to time (especially in later chapters) some higher mathematical facts will be employed, but knowledge of advanced mathematics is not required to follow most of the text. Because of the imposed mathematical limitations, some of the stated theorems need, say, additional measurability conditions to be completely precise. Also, less important (but nonignorable) technical conditions for some developments are sometimes omitted, but such developments are called "Results," rather than "Theorems."

The book is primarily concerned with discussing basic issues and principles of Bayesian analysis and decision theory. No systematic attempt is made to present a survey of actual developed methodology, i.e., to present specific developments of these ideas in particular areas of statistics. The examples that are given tend to be rather haphazard, and, unfortunately, do not cover some of the more difficult areas of statistics, such as nonparametrics. Nevertheless, a fair amount of methodology ends up being introduced, one way or another.

This second edition of the book has undergone a title change, with the addition of "Bayesian Analysis." This reflects the major change in the book, namely an extensive upgrading of the Bayesian material, to the point where the book can serve as a text on Bayesian analysis alone. The motivation for this upgrading was the realization that, although I professed to be a "rabid Bayesian" in the first edition (and still am), the first edition was not well suited for a primarily Bayesian course; in particular, it did not highlight the conditional Bayesian perspective properly. In attempting to correct this problem, I fell into the usual revision trap of being unable to resist adding substantial new material on subjects crucial to Bayesian analysis, such as hierarchical Bayes theory, Bayesian calculation, Bayesian communication, and combination of evidence.

For those familiar with the old book, the greatest changes are in Chapters 3 and 4, which were substantially enlarged and almost completely rewritten. Some sections of Chapter 1 were redone (particularly 1.6), and some small subsections were added to Chapter 2. The only significant change to Chapter 5 was the inclusion of an introduction to the now vast field of minimax multivariate estimation (Stein estimation); this has become by far the largest statistical area of development within minimax theory. Only very minor changes were made to Chapter 6, and Chapter 7 was changed only by the addition of a section discussing the issue of optional stopping. A number of changes were made to Chapter 8, in light of recent developments, but no thorough survey was attempted.

In general, no attempt was made to update references in parts of the book that were not rewritten. This, unfortunately, perpetuated a problem with the first edition, namely the lack of references to the early period of decision theory. Many of the decision-theoretic ideas and concepts seem to have become part of the folklore, and I apologize for not making the effort to trace them back to their origins and provide references.

In terms of teaching, the book can be used as a text for a variety of courses. The easiest such use is as a text in a two-semester or three-quarter course on Bayesian analysis and statistical decision theory; one can simply proceed through the book. (Chapters 1 through 4 should take the first semester, and Chapters 5 through 8 the second.) The following are outlines for various possible *single-semester* courses. The first outline is for a master's level course, and has a more applied orientation, while the other outlines also include theoretical material perhaps best suited for Ph.D. students. Of course, quite different arrangements could also be used successfully.

#### Bayesian Analysis and Decision Theory (Applied)

1 (except 1.4, 1.7, 1.8); 2; 3 (except 3.4, 3.5.5, 3.5.6, 3.5.7); 4 (except 4.4.4, 4.7.4 through 4.7.11, 4.8, 4.11); 7 (except 7.4.2 through 7.4.10, 7.5, 7.6); valuable other material to cover, if there is time, includes 4.7.4, 4.7.5, 4.7.9, 4.7.10, 4.7.11, and 4.11.

#### Bayesian Analysis and Decision Theory (More Theoretical)

1; 2 (except 2.3, 2.4.3, 2.4.4, 2.4.5); 3 (except 3.4, 3.5.5, 3.5.6, 3.5.7); 4 (except 4.4.4, 4.5.3, 4.6.3, 4.6.4, 4.7.4, 4.7.6, 4.7.7, 4.7.9, 4.7.10, 4.8.3, 4.9, 4.10, 4.11);

- (i) With Minimax Option: 5 (except 5.2.3); parts of 8.
- (ii) With Invariance Option: 6; parts of 8.
- (iii) With Sequential Option: 7 (except 7.4.7 through 7.4.10, 7.5.5, 7.6); parts of 8.



**A Mainly Bayesian Course (More Theoretical)**

1 (except 1.4, 1.8); 2 (except 2.3); 3 (except 3.5.5 and 3.5.6); 4 (except 4.7.6, 4.7.7); 7 (except 7.4.2 through 7.4.10, 7.5, 7.6); more sequential Bayes could be covered if some of the earlier sections were eliminated.

**A Mainly Decision Theory Course (Very Theoretical)**

1 (except 1.6); 2 (except 2.3); Sections 3.3, 4.1, 4.2, 4.4, 4.8; 5 (except 5.2.3); 6; 7 (except 7.2, 7.4, 7.7); 8.

I am very grateful to a number of people who contributed, in one way or another, to the book. Useful comments and discussion were received from many sources; particularly helpful were Eric Balder, Mark Berliner, Don Berry, Sudip Bose, Lawrence Brown, Arthur Cohen, Persi Diaconis, Roger Farrell, Leon Gleser, Bruce Hill, Tzou Wu-Jien Joe, T. C. Kao, Jack Kiefer, Sudhakar Kunte, Erich Lehmann, Carl Morris, Herman Rubin, S. Sivaganesan, Bill Studden, Don Wallace, Robert Wolpert, and Arnold Zellner. I am especially grateful to Herman Rubin: he provided most of the material in Subsections 7.4.8 and 7.4.9, and was my “foolishness filter” on much of the rest of the book.

The first edition of the book was typed by Lou Anne Scott, Norma Lucas, Kathy Woods, and Carolyn Knutsen, to all of whom I am very grateful. The highly trying job of typing this revision was undertaken by Norma Lucas, and her skill and cheer throughout the process were deeply appreciated. Finally, I would like to express my appreciation to the John Simon Guggenheim Memorial Foundation, the Alfred P. Sloan Foundation, and the National Science Foundation for support during the writing of the book.

West Lafayette, Indiana  
*March 1985*

JAMES BERGER

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