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Susanne C. Brenner L. Ridgway Scott

The Mathematical Theory of Finite Element Methods

Second Edition

With 41 Illustrations

Springer

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Series Editors

Cover illustration: Flow of a Newtonian fluid at Reynolds number 58, computed with the techniques developed in Chapters 12 and 13.

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Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series Texts in Applied Mathematics (TAM).

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and to encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the Applied Mathematical Sciences (AMS) series, which will focus on advanced textbooks and research-level monographs.

Pasadena, California — J.E. Marsden
Providence, Rhode Island — L. Sirovich
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Preface to the Second Edition

This edition contains two new chapters. The first one is on the additive Schwarz theory with applications to multilevel and domain decomposition preconditioners, and the second one is an introduction to *a posteriori* error estimators and adaptivity. We have also included a new section on an example of a one-dimensional adaptive mesh, a new section on the discrete Sobolev inequality and new exercises throughout. The list of references has also been expanded and updated.

We take this opportunity to extend thanks to everyone who provided comments and suggestions about this book over the years, and to the National Science Foundation for support. We also wish to thank Achi Dosanjh and the production staff at Springer-Verlag for their patience and care.

Columbia, SC
Chicago, IL
20/02/2002

Susanne C. Brenner
L. Ridgway Scott

Preface to the First Edition

This book develops the basic mathematical theory of the finite element method, the most widely used technique for engineering design and analysis. One purpose of this book is to formalize basic tools that are commonly used by researchers in the field but never published. It is intended primarily for mathematics graduate students and mathematically sophisticated engineers and scientists.

The book has been the basis for graduate-level courses at The University of Michigan, Penn State University and the University of Houston. The prerequisite is only a course in real variables, and even this has not been necessary for well-prepared engineers and scientists in many cases. The book can be used for a course that provides an introduction to basic functional analysis, approximation theory and numerical analysis, while building upon and applying basic techniques of real variable theory.

Chapters 0 through 5 form the essential material for a course. Chapter 0 provides a microcosm of what is to follow, developed in the one-dimensional case. Chapters 1 through 4 provide the basic theory, and Chapter 5 develops basic applications of this theory. From this point, courses can bifurcate in various directions. Chapter 6 provides an introduction to efficient iterative solvers for the linear systems of finite element equations. While essential from a practical point of view (our reason for placing it in a prominent position), this could be skipped, as it is not essential for further chapters. Similarly, Chapter 7, which derives error estimates in the maximum norm and shows how such estimates can be applied to nonlinear problems, can be skipped as desired.

Chapter 8, however, has an essential role in the following chapters. But one could cover only the first and third sections of this chapter and then go on to Chapter 9 in order to see an example of the more complex systems of differential equations that are the norm in applications. Chapter 10 depends to some extent on Chapter 9, and Chapter 11 is essentially a continuation of Chapter 10. Chapter 12 presents Banach space interpolation techniques with applications to convergence results for finite element methods. This is an independent topic at a somewhat more advanced level.

To be more precise, we describe three possible course paths that can be chosen. In all cases, the first step is to follow Chapters 0 through 5. Someone interested to present some of the "hard estimates" of the subject could then choose from Chapters 6 through 8, and 12. On the other hand, someone interested more in physical applications could select from Sect. 8.1, Sect. 8.3 and Chapters 9 through 11. Someone interested primarily in algorithmic efficiency and code development issues could follow Chapters 6, 8, 10 and 11.

The omissions from the book are so numerous that is hard to begin to list them. We attempt to list the most glaring omissions for which there are excellent books available to provide material.

We avoid time-dependent problems almost completely, partly because of the existence of the book of (Thomée 1984). Our extensive development of different types of elements and the corresponding approximation theory is complementary to Thomée's approach. Similarly, our development of physical applications is limited primarily to linear systems in continuum mechanics. More substantial physical applications can be found in the book by (Johnson 1987).

Very little is said here about adaptivity. This active research area is addressed in various conference proceedings (cf. Babuška, Chandra & Flaherty 1983 and Babuška, Zienkiewicz, Gago & de A. Oliveira 1986).

We emphasize the variety of discretizations (that is, different "elements") that can be used, and we present them (whenever possible) as families depending on a parameter (usually the degree of approximation). Thus, a spirit of "high-order" approximations is developed, although we do not consider increasing the degree of approximation (as is done in the so-called P-method and spectral element method) as the means of obtaining convergence. Rather, we focus on mesh subdivision as the convergence parameter. The recent book by (Szabo & Babuška 1991) may be consulted for alternatives in this direction.

Although we provide a brief introduction to mixed methods, the importance of this subject is not appropriately reflected here. However, the recent book by (Brezzi & Fortin 1991) can be consulted for a thorough treatment of the subject.

We draw extensively on the book of (Ciarlet 1978), both following many of its ideas and using it as a reference for further development of various subjects. This book has recently been updated in (Ciarlet & Lions 1991), which also contains an excellent survey of mixed methods. Moreover, the Handbook series to which the latter reference belongs can be expected to provide valuable reference material in the future.

We take this opportunity to thank the many people who have helped at various stages, and in many different ways, in the preparation of this book. The many students who struggled through early drafts of the book made invaluable contributions. Many readers of the preliminary versions will find their specific suggestions incorporated.

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