An Introduction to Minimax Theorems and Their Applications to Differential Equations

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An Introduction to Minimax Theorems and Their Applications to Differential Equations

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To our children

Isabel and Luis

and

Takuhi and Lusina

CONTENTS

PREFACE

This text is meant to be an introduction to critical point theory and its applications to differential equations. It is designed for graduate and postgraduate students as well as for specialists in the fields of differential equations, variational methods and optimization. Although related material can be found in other books, the treatment here has the following main purposes:

- To present a survey on existing minimax theorems,
- To give applications to elliptic differential equations in bounded domains and periodic second-order ordinary differential equations,
- To consider the dual variational method for problems with continuous and discontinuous nonlinearities,
- To present some elements of critical point theory for locally Lipschitz functionals and to give applications to fourth-order differential equations with discontinuous nonlinearities,
- To study homoclinic solutions of differential equations via the variational method.

The Contents of the book consist of seven chapters, each one divided into several sections. A bibliography is attached to the end of each chapter.

In Chapter I, we present minimization theorems and the mountain-pass theorem of Ambrosetti–Rabinowitz and some of its extensions. The concept of differentiability of mappings in Banach spaces, the Fréchet's and Gâteaux derivatives, second-order derivatives and general minimization theorems, variational principles of Ekeland [Ek1] and Borwein & Preiss [BP] are proved and relations to the minimization problem are given. Deformation lemmata, Palais–Smale conditions and mountain-pass theorems are considered. The deformation approach and the ε−variational approach are applied to prove the mountain-pass theorem and several extensions. We consider deformation theorems and Palais–Smale type conditions of Cerami $(PSC)_c$, Schechter $(PS)_{c,\psi}$, and $(PS)_c^*$-condition in scales of Banach spaces. We prove the mountain-pass theorem of Ambrosetti & Rabinowitz [ARa] and its extensions due to Cerami [Ce], Willem [Wil1], Pucci & Serin [PS1], Rabinowitz [Ra1], Schechter [Sch1], Brezis & Nirenberg [BN], Aubin & Ekeland [AE], Ghossoub & Preiss [GP]. A variant of a three critical point theorem with $(PS)_{c,\psi}$ condition is proved.

In Chapter II, we present saddle point theorems of Rabinowitz and its extensions due to Lazer & Solimini [LS] and Schechter [Sch2]. The concept of local linking and the three critical points theorem of Brezis & Nirenberg [BN], Li & Willem [LW] are presented. Linking theorems due to E.A. Silva [EAS] are also considered.

In Chapter III, we consider applications of critical point theorems to elliptic problems in bounded domains. We study the Neumann problem and Hammerstein equations on a bounded domain $\Omega \subset \mathbf{R}^n$, with smooth boundary Γ, and present some results obtained by applying variational methods. We characterize the range of the Neumann problem

$$\begin{aligned} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + g\,(u) &= f, \quad x \in \Omega, \\ \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_i}\cos\,(\nu, x_j) &= 0, \quad x \in \Gamma, \end{aligned}$$

where ν is the unit exterior normal to Γ and $a_{ij} \in C^1(\bar{\Omega})$, $a_{ij}(x) = a_{ji}(x)$.

Next we consider the problem of finding solutions $u \in L_2(\Omega)$ to the Hammerstein integral equation

$$u(t) = \int\limits_{\Omega} k(t,s)f(s,u(s))ds.$$

Under suitable conditions on the nonlinear function f and the kernel k, we prove existence results using the mountain-pass theorem of Ambrosetti–Rabinowitz. We prove existence of nontrivial solutions using the saddle-point theorem due to Lazer–Solimini.

In Chapter IV, we apply variational methods to prove the existence of periodic solutions of some second-order nonlinear differential equations, namely in resonance situations. More precisely, we consider the equation

$$u'' + \alpha\,(x)\,u + g(x,u) = 0,$$

where the function g will be assumed $2\pi-$periodic in x and with superlinear growth in u and α will be a bounded $2\pi-$periodic function. Situation of "resonance" or "non-uniform non-resonance" type will be considered. They are refered to the location of the function α with respect to the eigenvalues of the linear operator $Lu = u''$ with periodic conditions. We establish some existence results using mountain-pass theorem of Ambrosetti–Rabinowitz, a saddle-point theorem due to Silva and a linking theorem of Li–Willem.

In Chapter V, we present the dual variational method and its applications to some problems for fourth-order differential equations. Some preliminaries on convex functions and Fenchel–Legendre transform are presented.

Then, applications to problems for fourth-order differential equations with continuous and discountionuous nonlinearities are given.

The purpose of Chapter VI is to present several variants of minimization and mountain-pass theorems for nondifferentiable functionals. We assume that the given functionals are locally Lipschitz so that their generalized gradients can be defined (cf. Clarke [Cl]). We present a proof of a generalized mountain-pass theorem for locally Lipschitz functionals based on the Ekeland's variational principle. Consequently we obtain the mountain-pass theorems due to Chang [Ch1], Ghoussoub & Preiss [GP] and Brezis & Nirenberg [BN]. Further we introduce a variant of Palais–Smale condition. Again, using Ekeland's variational principle we prove minimization, coercivity and mountain-pass theorems. The abstract theorems are applied to problems for fourth-order differential equations with discontinuous nonlinearities.

In Chapter VII, we present several existence results for homoclinic solutions of differential equations via variational method. In recent years, starting with works of Bolotin [Bol], Coti-Zelati, Ekeland and Séré [CZES], Coti-Zelati & Rabinowitz [CZR1], [CZR2], Rabinowitz [Ra4], variational methods have been applied to study the existence of homoclinic and heteroclinic solutions of second-order equations and Hamiltonian systems. The search of homoclinic and heteroclinic solutions is a classical problem, originated in the work of Poincaré. We start Chapter VII with some preliminaries on dynamical systems.

We apply the variational method to prove existence of positive homoclinic solutions of a second-order equation.

Next we study the existence of homoclinic solutions of the fourth-order extended Fisher–Kolmogorov equation, that appears in several branches of Physics.

Further we consider homoclinic type solutions $u \in H^1(\mathbf{R}^n)$, $n \geq 3$ of Schrödinger equations on $\mathbf{R}^n$ of the form

$$-\Delta u + V(x)\,u = f(u), \quad x \in \mathbf{R}^n,$$

where V and f satisfy suitable conditions. We also consider Schrödinger type equations in a strip-like domain using variational method.

Finally, we generalize the approach considering nontrivial solutions of the semilinear problem

$$a(u,v) + \int_X V(x)\,u(x)\,v(x)\,m(dx) = \int_X f(u)\,v(x)\,m(dx),$$

where a is a strongly local, regular Dirichlet form on the topological Hausdorff space X, endowed with the positive Radon measure m. We recall a framework of homogeneous spaces, homogeneous dimension and the related Sobolev inequalities. As an example semilinear Kohn–Laplace equations in $\mathbf{R}^3$ is considered.

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