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(continued after index)

Pierre Brémaud

Markov Chains

Gibbs Fields, Monte Carlo Simulation, and Queues

With 64 Illustrations

Springer

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Mathematics Subject Classification (1991): 60-01, 93Exx

Library of Congress Cataloging-in-Publication Data
Brémaud, Pierre.
Markov chains : Gibbs fields, Monte Carlo simulation, and queues / Pierre Brémaud.
p. cm. – (Texts in applied mathematics ; 31)
Includes bibliographical references (p. –) and index.
ISBN 978-1-4419-3131-3 ISBN 978-1-4757-3124-8 (eBook)
DOI 10.1007/978-1-4757-3124-8
1. Markov processes. I. Title. II. Series.
QA274.8.B74 1998
519.233–dc21 98-17539

Printed on acid-free paper.

9 8 7 6 5 4

springeronline.com

To Marion

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics* (*TAM*).

The development of new courses is a natural consequence of high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences* (*AMS*) series, which will focus on advanced textbooks and research level monographs.

Preface

From Pushkin to Monte Carlo

When Markov introduced his famous model in 1906, he was not preoccupied with applications. He just wanted to show that independence is not necessary for the law of large numbers (the weak law of large numbers at that time, since Borel proved the strong law for heads and tails only in 1909). An example that he considered was the alternation of consonants and vowels in Pushkin's *Eugene Onegin*, which he described as a two-state chain. (This, however, does not say much about the plot!) Almost at the same time, and faithful to the French tradition of gambling probabilists, Poincaré studied Markov chains on finite groups, with applications to card shuffling. The Austrian physicists Paul and Tatiana Ehrenfest proposed in 1907 a Markov chain model that very much helped to clarify the controversial issue of thermodynamic irreversibility. Sir Francis Galton, a cousin of Darwin, who was interested in the probability of the survival of the English peerage, was the inventor of the branching process, another famous Markov model with many applications besides the original one. He posed the problem in the *Educational Times* in 1873, and in the same year and same journal, Reverend Watson proposed the method of solution that became a textbook classic.

The dates mentioned above show that Markov models were already around even before Markov started the systematic study of this class of random sequences. However, the work of Markov challenged the best probabilists, such as Kolmogorov, Doeblin, and Fréchet, just to mention the leading pioneering figures. The outcome was a clean and sound theory ready for applications, and today, Markov chains are omnipresent in the applied sciences. For instance, *biology* is an important consumer of Markov models, many of them concerning genetics and population theory. In the *social sciences*, social mobility can be described in Markovian terms. Quantitative *psychology* uses Markovian models of learning. *Physics* is a major patron of Markov chain theory, and Markov models (for instance, the Ehrenfest diffusion model, the annealing model, and the Ising–Peierls model of phase transition) have been very useful in understanding qualitatively complex phenomena. Markov chains have found many applications in *electrical engineering*, for instance in the performance analysis of multiple access communications protocols and of communications networks, in coded modulation, and in image processing. Recently, Markov chain theory has received an additional impetus from the advent of Monte Carlo Markov chain simulation. Markov chains have found a privileged domain of application in *operations research*, in reliability theory and queuing theory for instance.

A Useful, Simple, and Beautiful Theory

The list of applications of Markov chains is virtually infinite, and one is entitled to say that it is the single most successful class of stochastic processes, its success being due to the relative simplicity of its theory and to the fact that simple Markov models can exhibit extremely varied and complex behavior. The modeling power of Markov chains may well be compared to that of ordinary differential equations.

The theory of Markov chains with a countable state space is an ideal introduction to stochastic processes; it is not protected by a wall of technicalities, and therefore the student has quick access to the main results. Indeed, the mathematical equipment required for a rewarding study of this class of models consists only of the notion of conditional independence and the strong law of large numbers.

Another pleasant feature of Markov chain theory is that this classic topic can be presented in terms of the elegant concepts of the modern theory of stochastic processes, such as reversibility, martingales, and coupling.

Basic Theory and Advanced Topics

This book is devoted to the study of homogeneous Markov chains (HMCs) with a countable state space, in discrete time and in continuous time. The table of contents reflects the recent advances of the theory and responds to a growing need for a unified treatment of related topics such as finite Gibbs fields, nonhomogeneous Markov chains, discrete-time regenerative processes, Monte Carlo simulation, simulated annealing, and queuing theory.

About half of the book is devoted to the basic theory (Chapters 2, 3, 4, and 8). This part of the book introduces *discrete-time Markov models* (Chapter 2) and gives an account of their *recurrence and ergodicity* (Chapter 3) properties as well as of their *long-run behavior* (Chapter 4). It also treats the *continuous-time Markov models* (Chapter 8). Continuous-time HMCs are essentially discrete-time HMCs with a random time scale. The time separating two successive transitions is not one unit, but it is an exponential time depending on the current state. (This quick description, of course, jumps over the fine technical details.) Another informal statement is that continuous-time HMCs are discrete-time HMCs with a little dose of Poisson processes. We shall say no more at this point except that the traditional semigroup approach of Chapter 8 is completed in Chapter 9, on *Poisson calculus and queues*, by a probabilistic approach, where continuous-time HMCs are described by a family of independent Poisson processes associated with a deterministic rule selecting state transitions. This approach has many advantages, but it is not yet standard in textbooks, and besides, it requires some mathematical maturity. It is one of the advanced topics of this book (Chapters 5, 6, 7, and 9).

At least one of three requirements must be satisfied for a topic to be called advanced in this book. It has to be mathematically subtle, or technical, or simply not a textbook standard.

The chapter on *eigenvalues and nonhomogeneous Markov chains* (Chapter 6) is, for instance, a little technical, but it is important and concerns an area of intense research activity. It deals with the topic of effective computation of convergence rates, both for convergence to steady state of an ergodic chain and for convergence of ergodic estimates, in view of applications to simulation, for instance. It also includes the basic results of nonhomogeneous Markov chain theory, which will be applied to the analysis of the simulated annealing algorithm. The chapter on *Lyapunov functions and martingales* (Chapter 5) is more mathematical, and contains a brief introduction to potential theory. It gives the powerful theorem of Foster, a sufficient condition of positive recurrence, and places it in its natural environment among martingales and potentials. It is a first contact with martingale theory whose efficiency is demonstrated by a few examples concerning the absorption problem. *Gibbs fields and Monte Carlo simulation* (Chapter 7) are very important topics, of interest in physics, image processing, and optimization. There are quite a few good reasons to include Gibbs fields in a book devoted to Markov chains, besides the observation that they generalize in a natural way the Markov definition on the real line to arbitrary discrete index sets. One of them is that they are a privileged domain of application of the *Monte Carlo* Markov chain simulation algorithms. The latter is the last topic of Chapter 7, which also includes an introduction to simulated annealing.

Chapter 1 is a *probability review*, and it has a structure slightly different from the others in that it contains worked out exercises that the reader is expected to try a few minutes before looking at the solution. An *appendix* provides the results in number theory, analysis, and linear algebra directly useful to the theory of Markov chains. Each chapter is closed by a problems section.

Acknowledgments

My colleagues David Aldous, of the University of California at Berkeley, Jim Fill, of the Johns Hopkins University, Michael Frey, of the University of Ulm, Takis Konstantopoulos, of the University of Texas at Austin, Torgny Lindvall, of the Chalmers University of Technology and the Göteborg University, and Laurent Massoulié, of the Centre National d'Etudes en Télécommunications, helped me with their corrections and suggestions. My friends in the Laboratory of Signals and Systems of the Centre National de la Recherche Scientifique Hervé Carfantan, Philippe Ciuciu, Jean-François Giovannelli, and Jérôme Idier have supported me in many ways (LaTeX and Xfig, corrections, coffee break cookies).

To all of them I wish to express my sincere gratitude, as well as to Pam Williams, who diligently typed a first draft of the manuscript, and to my students at the Ecole Polytechnique, at the Swiss Federal Institute of Technology of Lausanne, and at the Ecole Nationale Supérieure de Techniques Avancées, who have contributed to many improvements of the manuscript.

Gif-sur-Yvette
November, 1998 *Pierre Brémaud*

Contents

Series Preface vii

Preface ix

1 Probability Review 1

1 Basic Concepts . . . 1
1.1 Events . . . 1
1.2 Random Variables . . . 3
1.3 Probability . . . 4
2 Independence and Conditional Probability . . . 7
2.1 Independence of Events and of Random Variables . . . 7
2.2 Bayes's Rules . . . 9
2.3 Markov Property . . . 11
3 Expectation . . . 14
3.1 Cumulative Distribution Function . . . 14
3.2 Expectation, Mean, and Variance . . . 15
3.3 Famous Random Variables . . . 20
4 Random Vectors . . . 25
4.1 Absolutely Continuous Random Vectors . . . 25
4.2 Discrete Random Vectors . . . 28
4.3 Product Formula for Expectation . . . 29
5 Transforms of Probability Distributions . . . 29
5.1 Generating Functions . . . 29
5.2 Characteristic Functions . . . 34
6 Transformations of Random Vectors . . . 36
6.1 Smooth Change of Variables . . . 36
6.2 Order Statistics . . . 38
7 Conditional Expectation of Discrete Variables . . . 39
7.1 Definition and Basic Properties . . . 39
7.2 Successive Conditioning . . . 41
8 The Strong Law of Large Numbers . . . 42
8.1 Borel–Cantelli Lemma . . . 42
8.2 Almost-Sure Convergence . . . 43

8.3 Markov's Inequality 45
8.4 Proof of Kolmogorov's SLLN 47

2 Discrete-Time Markov Models **53**
1 The Transition Matrix 53
1.1 Markov Property 53
1.2 Distribution of an HMC 56
2 Markov Recurrences 58
2.1 A Canonical Representation 58
2.2 A Few Famous Examples 59
3 First-Step Analysis 65
3.1 Absorption Probability 65
3.2 Mean Time to Absorption 68
4 Topology of the Transition Matrix 71
4.1 Communication 71
4.2 Period 72
5 Steady State 75
5.1 Stationarity 75
5.2 Examples 76
6 Time Reversal 80
6.1 Reversed Chain 80
6.2 Time Reversibility 81
7 Regeneration 83
7.1 Strong Markov Property 83
7.2 Regenerative Cycles 86

3 Recurrence and Ergodicity **95**
1 Potential Matrix Criterion 95
1.1 Recurrent and Transient States 95
1.2 Potential Matrix 97
1.3 Structure of the Transition Matrix 100
2 Recurrence and Invariant Measures 100
3 Positive Recurrence 104
3.1 Stationary Distribution Criterion 104
3.2 Examples 105
4 Empirical Averages 110
4.1 Ergodic Theorem 110
4.2 Examples 113
4.3 Renewal Reward Theorem 117

4 Long Run Behavior **125**
1 Coupling 125
1.1 Convergence in Variation 125
1.2 The Coupling Method 128

2 Convergence to Steady State 130
2.1 Positive Recurrent Case 130
2.2 Null Recurrent Case 131
2.3 Thermodynamic Irreversibility 133
2.4 Convergence Rates via Coupling 136
3 Discrete-Time Renewal Theory 137
3.1 Renewal Equation 137
3.2 Renewal Theorem 140
3.3 Defective Renewal Sequences 142
4 Regenerative Processes 145
4.1 Renewal Equation of a Regenerative Process 145
4.2 Regenerative Theorem 146
5 Life Before Absorption 149
5.1 Infinite Sojourns 149
5.2 Time to Absorption 153
6 Absorption 154
6.1 Fundamental Matrix 154
6.2 Absorption Matrix 156

5 Lyapunov Functions and Martingales **167**
1 Lyapunov Functions 167
1.1 Foster's Theorem 167
1.2 Queuing Applications 173
2 Martingales and Potentials 178
2.1 Harmonic Functions and Martingales 178
2.2 The Maximum Principle 180
3 Applications of Martingales to HMCs 185
3.1 The Two Pillars of Martingale Theory 185
3.2 Transience and Recurrence via Martingales 186
3.3 Absorption via Martingales 189

6 Eigenvalues and Nonhomogeneous Markov Chains **195**
1 Finite Transition Matrices 195
1.1 Perron–Frobenius Theorem 195
1.2 Quasi-stationary Distributions 199
2 Reversible Transition Matrices 201
2.1 Eigenstructure and Diagonalization 201
2.2 Spectral Theorem 204
3 Convergence Bounds Without Eigenvectors 207
3.1 Basic Bounds, Reversible Case 207
3.2 Nonreversible Case 211
4 Geometric Bounds 212
4.1 Weighted Paths 212
4.2 Conductance 215

5 Probabilistic Bounds 219
5.1 Separation and Strong Stationary Times 219
5.2 Convergence Rates via Strong Stationary Times 223
6 Fundamental Matrix of Recurrent Chains 226
6.1 Definition of the Fundamental Matrix 226
6.2 Mutual Time–Distance Matrix 230
6.3 Variance of Ergodic Estimates 232
7 The Ergodic Coefficient 235
7.1 Dobrushin's Inequality 235
7.2 Interaction Coefficients and Coincidence 238
8 Nonhomogeneous Markov Chains 239
8.1 Ergodicity of Nonhomogeneous Markov Chains 239
8.2 Block Criterion of Weak Ergodicity 241
8.3 Sufficient Condition of Strong Ergodicity 242

7 Gibbs Fields and Monte Carlo Simulation **253**
1 Markov Random Fields 253
1.1 Neighborhoods and Local Specification 253
1.2 Cliques, Potential, and Gibbs Distributions 256
2 Gibbs–Markov Equivalence 260
2.1 From the Potential to the Local Specification 260
2.2 From the Local Specification to the Potential 261
3 Image Models 268
3.1 Textures 268
3.2 Lines and Points 270
4 Bayesian Restoration of Images 275
4.1 MAP Likelihood Estimation 275
4.2 Penalty Methods 279
5 Phase Transitions 280
5.1 Spontaneous Magnetization 280
5.2 Peierls's Argument 281
6 Gibbs Sampler 285
6.1 Simulation of Random Fields 285
6.2 Convergence Rate of the Gibbs Sampler 288
7 Monte Carlo Markov Chain Simulation 290
7.1 General Principle 290
7.2 Convergence Rates in MCMC 295
7.3 Variance of Monte Carlo Estimators 299
8 Simulated Annealing 305
8.1 Stochastic Descent and Cooling 305
8.2 Convergence of Simulated Annealing 311

8 Continuous-Time Markov Models **323**
1 Poisson Processes 323
1.1 Point Processes 323
1.2 Counting Process of an HPP 324
1.3 Competing Poisson Processes 327
2 Distribution of a Continuous-Time HMC 329
2.1 Transition Semigroup 329
2.2 Infinitesimal Generator 333
3 Kolmogorov's Differential Systems 338
3.1 Finite State Space 338
3.2 General Case 340
3.3 Regular Jumps 344
4 The Regenerative Structure 345
4.1 Strong Markov Property 345
4.2 Embedded Chain and Transition Times 348
4.3 Explosions 350
5 Recurrence 357
5.1 Stationary Distribution Criterion of Ergodicity 357
5.2 Time Reversal 361
6 Long-Run Behavior 363
6.1 Ergodic Chains 363
6.2 Absorbing Chains 364

9 Poisson Calculus and Queues **369**
1 Continuous-Time Markov Chains as Poisson Systems 369
1.1 Strong Markov Property of HPPs 369
1.2 From Generator to Markov Chain 372
2 Stochastic Calculus of Poisson Processes 375
2.1 Counting Integrals and the Smoothing Formula 375
2.2 Kolmogorov's Forward System via Poisson Calculus 378
2.3 Watanabe's Characterization of Poisson Processes 380
3 Poisson Systems 383
3.1 The Purely Poissonian Description 383
3.2 The GSMP construction 385
3.3 Markovian Queues as Poisson Systems 388
4 Markovian Queuing Theory 394
4.1 Isolated Markovian Queues 394
4.2 The M/GI/1/∞/FIFO Queue 398
4.3 The GI/M/1/∞/FIFO Queue 402
4.4 Markovian Queuing Networks 407

Appendix 417
1 Number Theory and Calculus . . . 417
1.1 Greatest Common Divisor . . . 417
1.2 Abel's Theorem . . . 418
1.3 Lebesgue's Theorems for Series . . . 420
1.4 Infinite Products . . . 422
1.5 Tychonov's Theorem . . . 423
1.6 Subadditive Functions . . . 423
2 Linear Algebra . . . 424
2.1 Eigenvalues and Eigenvectors . . . 424
2.2 Exponential of a Matrix . . . 426
2.3 Gershgorin's Bound . . . 427
3 Probability . . . 428
3.1 Expectation Revisited . . . 428
3.2 Lebesgue's Theorems for Expectation . . . 430

Bibliography 433

Author Index 439

Subject Index 441