

# Practical Bilevel Optimization: Algorithms and Applications

# Nonconvex Optimization and Its Applications

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Volume 30

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# Practical Bilevel Optimization

*Algorithms and Applications*

by

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## PREFACE

The use of optimization techniques has become integral to the design and analysis of most industrial and socio-economic systems. Great strides have been made recently in the solution of large-scale problems arising in such areas as production planning, airline scheduling, government regulation, and engineering design, to name a few. Analysts have found, however, that standard mathematical programming models are often inadequate in these situations because more than a single objective function and a single decision maker are involved. Multiple objective programming deals with the extension of optimization techniques to account for several objective functions, while game theory deals with the inter-personal dynamics surrounding conflict. Bilevel programming, the focus of this book, is in a narrow sense the combination of the two. It addresses the problem in which two decision makers, each with their individual objectives, act and react in a noncooperative, sequential manner. The actions of one affect the choices and payoffs available to the other but neither player can completely dominate the other in the traditional sense.

Over the last 20 years there has been a steady growth in research related to theory and solution methodologies for bilevel programming. This interest stems from the inherent complexity and consequent challenge of the underlying mathematics, as well as the applicability of the bilevel model to many real-world situations. The primary aim of this book is to provide a historical perspective on algorithmic development and to highlight those implementations that have proven to be the most efficient in their class. This aim, however, applies mainly to the linear version of the problem since there have been only a handful of algorithms developed for the nonlinear and discrete cases. Claims of relative efficiency would be problematic for those implementations. A corollary aim of the book is to provide a sampling of applications in order to demonstrate the versatility of the basic model and the limitations of current technology.

Prior to undertaking this project, I was involved with Professor Kiyotaka Shimizu of Keio University and Professor Yo Ishizuka of Sophia University in writing the text *Nondifferentiable and Two-Level Mathematical Programming*. As the title implies, the book treats the general area of two-level programming (as distinguished therein from bilevel programming), and contains a comprehensive discussion of nondifferentiable optimization. The final product was highly theoretical, aimed primarily at the research community. Only two chapters dealt specifically with the bilevel programming problem or what is sometimes referred to as the Stackelberg game. The need

for a greater discussion of algorithms and applications became apparent soon after publication. I felt that an unfulfilled demand still existed for a reference text that offered an integrated treatment of the field from the practitioner's point of view. This was the motivation for the sequel.

The intended audience here includes management scientists, operations researchers, industrial engineers, mathematicians and economists. Students with a background in deterministic operations research methods should be on solid ground in reading the text. Nevertheless, anyone with training in mathematics at a level found in a typical undergraduate engineering or science curricula should be able to handle most of the material. To facilitate those new to optimization, I have provided several chapters on mathematical programming. This material constitutes the first part of the book. It is written in sufficient detail to permit its use in a first-year graduate course on bilevel programming, with perhaps the first third of the course geared towards basic optimization.

In writing a text of this scope, it is impossible to thank all those who have helped or contributed in a significant way. A few individuals, though, should be singled out for special recognition. The first is Jim Falk who put me in touch with bilevel programming during my graduate days at The George Washington University, and whose basic research in nonlinear optimization has provided the foundations for many of the developments to date. Next there are Professors Shimizu and Ishizuka who were collaborators on my first bilevel text, and Jim Moore and Tom Edmunds who were two of my most prolific Ph.D. students. Their research is at the core of several of the most successful bilevel codes. I am also indebted to Gilles Savard and Luis Vicente for their many insights, and for giving me permission to use their work in several chapters. Similar thanks go to David Boyce, G. Anandalingam, Charles Macal, Patrice Marcotte, Joaquim Júdice, Jiming Liu and Leon Lasdon. In addition, there are those whose research I have cited repeatedly and who have made notable contributions to the field, including Mark Karwan, Wayne Bialis, A. Faustino, Pierre Hansen, Brigitte Jaumard, Omar Ben-Ayed and Charles Blair.

Finally, I wish to express my gratitude to the Department of Mechanical Engineering at the University of Texas where I have been a faculty member since 1984. If I had been asked to define a more supportive academic environment before coming to Texas, I am not sure what it would have been. I am deeply indebted to my colleagues on the faculty and to the graduate students with whom I have worked and taught over the years for their continued concern and support.

J. F. Bard