

**Division Algebras:  
Octonions,  
Quaternions,  
Complex Numbers  
and the Algebraic Design of Physics**

# Mathematics and Its Applications

---

Managing Editor:

**M. HAZEWINKEL**

*Centre for Mathematics and Computer Science, Amsterdam, The Netherlands*

---

Volume 290

---

# Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics

*by*

Geoffrey M. Dixon  
*Brandeis University, U.S.A.*



Springer-Science+Business Media, B.V.

## Library of Congress Cataloging-in-Publication Data

Dixon, Geoffrey M.

Division algebras : octonions, quaternions, complex numbers, and the algebraic design of physics / by Geoffrey M. Dixon.

p. cm. -- (Mathematics and its applications ; v. 290)  
Includes index.

1. Algebra. 2. Mathematical physics. I. Title. II. Series:  
Mathematics and its applications (Kluwer Academic Publishers) ; v.  
290.

QC20.7.A4D59 1994  
512'.57--dc20

94-13948

ISBN 978-1-4419-4746-8 ISBN 978-1-4757-2315-1 (eBook)

DOI 10.1007/978-1-4757-2315-1

---

*Printed on acid-free paper*

This printing is a digital duplication of the original edition.

All Rights Reserved

© 1994 Springer Science+Business Media Dordrecht. Second Printing 2002.

Originally published by Kluwer Academic Publishers in 1994

Softcover reprint of the hardcover 1st edition 1994

No part of the material protected by this copyright notice may be reproduced or utilized in any form or by any means, electronic or mechanical, including photocopying, recording or by any information storage and retrieval system, without written permission from the copyright owner.

# Table of Contents

|   |    |
|---|----|
| <b>I Underpinnings</b>                                  | 1  |
| 1.1 The Argument  | 1  |
| Screed I  | 1  |
| Screed II   | 3  |
| Screed III  | 7  |
| 1.2 Clifford Algebras                                   | 11 |
| Clifford Algebra $\mathbf{R}_{3,1}$                     | 16 |
| Pauli Algebra   | 18 |
| Clifford Algebra $\mathbf{R}_{1,3}$                     | 19 |
| Clifford Algebra $\mathbf{R}_{1,9}$                     | 20 |
| 1.3 Conjugations and Spinors                            | 22 |
| Nilpotent Clifford Algebras                             | 24 |
| Symplectic Nilpotent Clifford Algebras                  | 25 |
| 1.4 Algebraic Fundamentals of the Standard Model        | 29 |
| <br>  |    |
| <b>II Division Algebras Alone</b>                       | 31 |
| 2.1 Mostly Octonions                                    | 31 |
| 2.2 Adjoint Algebras                                    | 35 |
| 2.3 Clifford Algebras, Spinors                          | 40 |
| 2.4 Resolving the Identity of $\mathbf{O}_L$            | 43 |
| 2.5 Lie Algebras, Lie Groups, from $\mathbf{O}_L$       | 46 |
| 2.6 From Galois Fields to Division Algebras: An Insight | 49 |
| <br>  |    |
| <b>III Tensor Algebras</b>                              | 59 |
| 3.1 Tensoring Two: Clifford Algebras and Spinors        | 59 |
| 3.2 Tensoring Two: Spinor Inner Product                 | 61 |
| Link to Internal Symmetry                               | 64 |

|  |            |
|--|------------|
| 3.3 Tensoring Three: Clifford Algebras and Spinors | 66         |
| 3.4 Tensoring Three: Spinor Inner Product          | 68         |
| Resolving the Identity of $\mathbf{T}$             | 68         |
| The Trace of $X$                                   | 70         |
| 3.5 Derivation of the Standard Symmetry            | 73         |
| 3.6 $SU(2) \times SU(3)$ Multiplets, and $U(1)$    | 78         |
| $U(1)$ Charges                                     | 81         |
| <b>IV Connecting to Physics</b>                    | <b>83</b>  |
| 4.1 Connecting to Geometry                         | 83         |
| Dimensional Reduction                              | 85         |
| 4.2 Connecting to Particles                        | 90         |
| 4.3 Parity Nonconservation                         | 94         |
| Righthanded Dirac Operator                         | 94         |
| Lefthanded Dirac Operator                          | 95         |
| Full Parity Violating Dirac Operator               | 96         |
| 4.4 Gauge Fields                                   | 97         |
| 4.5 Weak Mixing                                    | 100        |
| 4.6 Gauging $SU(3)$                                | 105        |
| <b>V Spontaneous Symmetry Breaking</b>             | <b>109</b> |
| 5.1 Scalar Fields                                  | 109        |
| 5.2 Scalar Lagrangians                             | 111        |
| 5.3 Fermions and Scalars                           | 115        |
| <b>VI 10 Dimensions</b>                            | <b>117</b> |
| 6.1 Fermion Lagrangian                             | 117        |
| Matter/Antimatter Mixing                           | 120        |
| 6.2 More $SU(3)$                                   | 122        |
| 6.3 Freedom from Matter-Antimatter Mixing          | 124        |
| 6.4 (1,9)-Scalar Lagrangian                        | 126        |
| 6.5 Charge Conjugation on $\mathbf{T}_L(2)$        | 128        |
| 6.6 Charge Conjugation on $\mathbf{T}^2$           | 130        |
| The Meaning of Majorana                            | 132        |
| 6.7 10 Other Dimensions                            | 133        |
| The Clifford Algebra                               | 137        |

|  |     |
|--|-----|
| <b>VII Doorways</b>  | 141 |
| 7.1 Moufang and Other Identities   | 141 |
| Two Identities   | 145 |
| The Moufang Identities   | 148 |
| 7.2 Spheres and Lie Algebras   | 150 |
| $S^3$  | 151 |
| $S^7$  | 153 |
| Sphere Fibrations  | 159 |
| 7.3 Triality   | 160 |
| Triality Representations of $so(8)$  | 164 |
| The <i>Tri</i> in Triality   | 165 |
| Freudenthal's Principle of Triality  | 169 |
| 7.4 $LG_2$ and <i>Tri</i>  | 170 |
| $LG_2$ Triality Triplet  | 171 |
| 7.5 $LG_2$ Triplets and the $X$ -Product                                   | 175 |
| $LG_2^X$ General Solution  | 180 |
| $LG_2^X$ and the $X$ -Adjoint Algebra $\mathbf{O}_{LX}$                    | 187 |
| <br>   |     |
| <b>VIII Corridors</b>  | 191 |
| 8.1 Magic Square   | 191 |
| 8.2 The Ten $MS_{KK'}$   | 192 |
| 8.3 <i>Spinor</i> $_{KK'}$ Outer Products                                  | 198 |
| $\mathbf{C}$ Outer Product   | 198 |
| $\mathbf{Q}$ Outer Products  | 200 |
| $\mathbf{O}$ Outer Products  | 202 |
| 8.4 $LF_4 \simeq MS_{RO}$  | 203 |
| 8.5 $J_3^O$ and $F_4$  | 208 |
| 8.6 More Magic Square  | 213 |
| <br>   |     |
| Appendix i. $\mathbf{O}_L$ Actions: Product Rule $e_a e_{a+1} = e_{a+5}$   | 217 |
| Appendix ii. $\mathbf{O}_R$ Actions: Product Rule $e_a e_{a+1} = e_{a+5}$  | 221 |
| Appendix iii. $\mathbf{O}_L$ Actions: Product Rule $e_a e_{a+1} = e_{a+3}$ | 225 |
| Appendix iv. $\mathbf{O}_R$ Actions: Product Rule $e_a e_{a+1} = e_{a+3}$  | 229 |
| <br>   |     |
| Bibliography   | 233 |
| Index  | 235 |

# Preface

I don't know who Gigerenzer is, but he wrote something very clever that I saw quoted in a popular glossy magazine:

"Evolution has tuned the way we think to frequencies of co-occurrences, as with the hunter who remembers the area where he has had the most success killing game."

This sanguine thought explains my obsession with the division algebras. Every effort I have ever made to connect them to physics - to the design of reality - has succeeded, with my expectations often surpassed. Doubtless this strong statement is colored by a selective memory, but the kind of game I sought, and still seek, seems to frowst about this particular watering hole in droves. I settled down there some years ago and have never felt like leaving.

This book is about the beasts I selected for attention (if you will, to render this metaphor politically correct, let's say I was a nature photographer), and the kind of tools I had to develop to get the kind of shots I wanted (the tools that I found there were for my taste overly abstract and theoretical). Half of this book is about these tools, and some applications thereof that should demonstrate their power. The rest is devoted to a demonstration of the intimate connection between the mathematics of the division algebras and the Standard Model of quarks and leptons with  $U(1) \times SU(2) \times SU(3)$  gauge fields, and the connection of this model to 10-dimensional spacetime implied by the mathematics.

If you understand what I have written to this point, then there is a good chance you have the background to understand this book. As to agreeing with its philosophical premises, you probably won't. I've never yet found two mathematicians or physicists who agree on the connection between mathematics and physics, and its depth. However, such agreement is not required to appreciate the material presented.



I owe thanks to several people: to my wife, Suzanne Young of Harvard, whose work in archaeometry compelled me in some weird way to write the book; to my thesis advisor, Hugh Pendleton of Brandeis, for his advice, interest, and forbearance; and to Rafal Ablamowicz, Martin Cederwall, Pertti Lounesto, Corinne Manogue, Ian Porteous, and Tony Smith, all of whom at my instigation converged on Göteborg, Sweden, this last January (1994) to talk about division algebras, Clifford algebras, and physics. Each played a crucial role at this meeting, and the breadth and depth of their insights has greatly enriched this monograph.

Geoffrey Dixon, Waltham, Massachusetts, 21 February 1994.