

Texts in Applied Mathematics **7**

Editors

J.E. Marsden

L. Sirovich

M. Golubitsky

W. Jäger

F. John (*deceased*)

Advisor

G. Iooss

Springer

New York

Berlin

Heidelberg

Barcelona

Budapest

Hong Kong

London

Milan

Paris

Santa Clara

Singapore

Tokyo

Texts in Applied Mathematics

1. *Sirovich*: Introduction to Applied Mathematics.
2. *Wiggins*: Introduction to Applied Nonlinear Dynamical Systems and Chaos.
3. *Hale/Koçak*: Dynamics and Bifurcations.
4. *Chorin/Marsden*: A Mathematical Introduction to Fluid Mechanics, 3rd ed.
5. *Hubbard/West*: Differential Equations: A Dynamical Systems Approach: Ordinary Differential Equations.
6. *Sontag*: Mathematical Control Theory: Deterministic Finite Dimensional Systems.
7. *Perko*: Differential Equations and Dynamical Systems, 2nd ed.
8. *Seaborn*: Hypergeometric Functions and Their Applications.
9. *Pipkin*: A Course on Integral Equations.
10. *Hoppensteadt/Peskin*: Mathematics in Medicine and the Life Sciences.
11. *Braun*: Differential Equations and Their Applications, 4th ed.
12. *Stoer/Bulirsch*: Introduction to Numerical Analysis, 2nd ed.
13. *Renardy/Rogers*: A First Graduate Course in Partial Differential Equations.
14. *Banks*: Growth and Diffusion Phenomena: Mathematical Frameworks and Applications.
15. *Brenner/Scott*: The Mathematical Theory of Finite Element Methods.
16. *Van de Velde*: Concurrent Scientific Computing.
17. *Marsden/Ratiu*: Introduction to Mechanics and Symmetry.
18. *Hubbard/West*: Differential Equations: A Dynamical Systems Approach: Higher-Dimensional Systems.
19. *Kaplan/Glass*: Understanding Nonlinear Dynamics.
20. *Holmes*: Introduction to Perturbation Methods.
21. *Curtain/Zwart*: An Introduction to Infinite-Dimensional Linear Systems Theory.
22. *Thomas*: Numerical Partial Differential Equations: Finite Difference Methods.
23. *Taylor*: Partial Differential Equations: Basic Theory.
24. *Merkin*: Introduction to the Theory of Stability.

Lawrence Perko

Differential Equations and Dynamical Systems

Second Edition

With 224 Illustrations



Springer

Lawrence Perko
Department of Mathematics
Northern Arizona University
Flagstaff, AZ 86011
USA

Series Editors

Jerrold E. Marsden
Control and Dynamical Systems, 104-44
California Institute of Technology
Pasadena, CA 91125
USA

L. Sirovich
Division of Applied Mathematics
Brown University
Providence, RI 02912
USA

M. Golubitsky
Department of Mathematics
University of Houston
Houston, TX 77204-3476
USA

W. Jäger
Department of Applied Mathematics
Universität Heidelberg
Im Neuenheimer Feld 294
69120 Heidelberg, Germany

Mathematics Subject Classification (1991): 34A34, 34C35, 58F06, 58F14, 58F21, 58F25, 70K10

Library of Congress Cataloging-in-Publication Data

Perko, Lawrence.

Differential equations and dynamical systems / Lawrence Perko. —
2nd. ed.

p. cm. — (Texts in applied mathematics ; 7)

Includes bibliographical references and index.

1. Differential equations, Nonlinear. 2. Differentiable dynamical
systems. I. Title. II. Series.

QA372.P47 1996

515'.353—dc20

96-15204

Printed on acid-free paper.

© 1996, 1991 Springer-Verlag New York, Inc.

Softcover reprint of the hardcover 2nd 1996

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Act, may accordingly be used freely by anyone.

Production managed by Bill Imbornoni; manufacturing supervised by Johanna Tschebull.
Camera-ready copy prepared from the author's LaTeX files.

9 8 7 6 5 4 3 2 1

ISBN-13: 978-1-4684-0251-3 e-ISBN-13: 978-1-4684-0249-0
DOI: 10.1007/978-1-4684-0249-0

To my wife, Kathy, and children, Mary, Mike, Vince, Jenny and John, for all the joy they bring to my life.

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM)*.

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research level monographs.

Preface to the Second Edition

This book covers those topics necessary for a clear understanding of the qualitative theory of ordinary differential equations and the concept of a dynamical system. It is written for advanced undergraduates and for beginning graduate students. It begins with a study of linear systems of ordinary differential equations, a topic already familiar to the student who has completed a first course in differential equations. An efficient method for solving any linear system of ordinary differential equations is presented in Chapter 1.

The major part of this book is devoted to a study of nonlinear systems of ordinary differential equations and dynamical systems. Since most nonlinear differential equations cannot be solved, this book focuses on the qualitative or geometrical theory of nonlinear systems of differential equations originated by Henri Poincaré in his work on differential equations at the end of the nineteenth century as well as on the functional properties inherent in the solution set of a system of nonlinear differential equations embodied in the more recent concept of a dynamical system. Our primary goal is to describe the qualitative behavior of the solution set of a given system of differential equations including the invariant sets and limiting behavior of the dynamical system or flow defined by the system of differential equations. In order to achieve this goal, it is first necessary to develop the local theory for nonlinear systems. This is done in Chapter 2 which includes the fundamental local existence–uniqueness theorem, the Hartman–Grobman Theorem and the Stable Manifold Theorem. These latter two theorems establish that the qualitative behavior of the solution set of a nonlinear system of ordinary differential equations near an equilibrium point is typically the same as the qualitative behavior of the solution set of the corresponding linearized system near the equilibrium point.

After developing the local theory, we turn to the global theory in Chapter 3. This includes a study of limit sets of trajectories and the behavior of trajectories at infinity. Some unresolved problems of current research interest are also presented in Chapter 3. For example, the Poincaré–Bendixson Theorem, established in Chapter 3, describes the limit sets of trajectories of two-dimensional systems; however, the limit sets of trajectories of three-dimensional (and higher dimensional) systems can be much more complicated and establishing the nature of these limit sets is a topic of current research interest in mathematics. In particular, higher dimensional systems

can exhibit strange attractors and chaotic dynamics. All of the preliminary material necessary for studying these more advanced topics is contained in this textbook. This book can therefore serve as a springboard for those students interested in continuing their study of ordinary differential equations and dynamical systems and doing research in these areas. Chapter 3 ends with a technique for constructing the global phase portrait of a dynamical system. The global phase portrait describes the qualitative behavior of the solution set for all time. In general, this is as close as we can come to “solving” nonlinear systems.

In Chapter 4, we study systems of differential equations depending on parameters. The question of particular interest is: For what parameter values does the global phase portrait of a dynamical system change its qualitative structure? The answer to this question forms the subject matter of bifurcation theory. An introduction to bifurcation theory is presented in Chapter 4 where we discuss bifurcations at nonhyperbolic equilibrium points and periodic orbits as well as Hopf bifurcations. Chapter 4 ends with a discussion of homoclinic loop and Takens–Bogdanov bifurcations for planar systems and an introduction to tangential homoclinic bifurcations and the resulting chaotic dynamics that can occur in higher dimensional systems.

The prerequisites for studying differential equations and dynamical systems using this book are courses in linear algebra and real analysis. For example, the student should know how to find the eigenvalues and eigenvectors of a linear transformation represented by a square matrix and should be familiar with the notion of uniform convergence and related concepts. In using this book, the author hopes that the student will develop an appreciation for just how useful the concepts of linear algebra, real analysis and geometry are in developing the theory of ordinary differential equations and dynamical systems. The heart of the geometrical theory of nonlinear differential equations is contained in Chapters 2–4 of this book and in order to cover the main ideas in those chapters in a one semester course, it is necessary to cover Chapter 1 as quickly as possible.

Several new sections have been added in the second edition of this book. Sections 2.12 and 2.13 on center manifold theory and normal forms, which generalize the material in Section 2.11 for planar systems, are new. A more detailed development of Melnikov’s method, including some new theorems and their proofs and a new section on higher order Melnikov theory, is given in Chapter 4. A discussion of the codimension and the universal unfolding of a bifurcation as well as some examples of higher codimension bifurcations, such as the Takens–Bogdanov bifurcation, have also been added in Chapter 4. The book ends with a new section on one of the author’s favorite research topics, a study of the bifurcations that occur in the class of bounded quadratic systems. The student is encouraged to participate in this research project by determining the bifurcation surfaces that occur in the class of bounded quadratic systems in the problem set at the end of Chapter 4.

I would like to express my sincere appreciation to my colleagues Terrence Blows and Jim Swift for their many helpful suggestions which substantially improved this book. I would also like to thank Louella Holter for her patience and precision in typing the original manuscript.

Contents

Series Preface	vii
Preface to the Second Edition	ix
1 Linear Systems	1
1.1 Uncoupled Linear Systems	1
1.2 Diagonalization	6
1.3 Exponentials of Operators	10
1.4 The Fundamental Theorem for Linear Systems	16
1.5 Linear Systems in \mathbf{R}^2	20
1.6 Complex Eigenvalues	28
1.7 Multiple Eigenvalues	32
1.8 Jordan Forms	39
1.9 Stability Theory	51
1.10 Nonhomogeneous Linear Systems	60
2 Nonlinear Systems: Local Theory	65
2.1 Some Preliminary Concepts and Definitions	65
2.2 The Fundamental Existence-Uniqueness Theorem	70
2.3 Dependence on Initial Conditions and Parameters	79
2.4 The Maximal Interval of Existence	87
2.5 The Flow Defined by a Differential Equation	95
2.6 Linearization	101
2.7 The Stable Manifold Theorem	105
2.8 The Hartman–Grobman Theorem	119
2.9 Stability and Liapunov Functions	129
2.10 Saddles, Nodes, Foci and Centers	136
2.11 Nonhyperbolic Critical Points in \mathbf{R}^2	146
2.12 Center Manifold Theory	153
2.13 Normal Form Theory	162
2.14 Gradient and Hamiltonian Systems	169

3	Nonlinear Systems: Global Theory	179
3.1	Dynamical Systems and Global Existence Theorems	180
3.2	Limit Sets and Attractors	190
3.3	Periodic Orbits, Limit Cycles and Separatrix Cycles	200
3.4	The Poincaré Map	209
3.5	The Stable Manifold Theorem for Periodic Orbits	218
3.6	Hamiltonian Systems with Two Degrees of Freedom	232
3.7	The Poincaré–Bendixson Theory in \mathbf{R}^2	242
3.8	Lienard Systems	250
3.9	Bendixson’s Criteria	261
3.10	The Poincaré Sphere and the Behavior at Infinity	264
3.11	Global Phase Portraits and Separatrix Configurations	290
3.12	Index Theory	294
4	Nonlinear Systems: Bifurcation Theory	311
4.1	Structural Stability and Peixoto’s Theorem	312
4.2	Bifurcations at Nonhyperbolic Equilibrium Points	325
4.3	Higher Codimension Bifurcations at Nonhyperbolic Equilibrium Points	334
4.4	Hopf Bifurcations and Bifurcations of Limit Cycles from a Multiple Focus	341
4.5	Bifurcations at Nonhyperbolic Periodic Orbits	351
4.6	One-Parameter Families of Rotated Vector Fields	373
4.7	The Global Behavior of One-Parameter Families of Periodic Orbits	385
4.8	Homoclinic Bifurcations	390
4.9	Melnikov’s Method	405
4.10	Global Bifurcations of Systems in \mathbf{R}^2	421
4.11	Second and Higher Order Melnikov Theory	442
4.12	The Takens–Bogdanov Bifurcation	456
4.13	Coppel’s Problem for Bounded Quadratic Systems	466
	References	507
	Index	513