Undergraduate Texts in Mathematics

James G. Simmonds

A Brief on Tensor Analysis

With 28 Illustrations

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To my father,

My first and greatest teacher

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Preface

When I was an undergraduate, working as a co-op student at North American Aviation, I tried to learn something about tensors. In the Aeronautical Engineering Department at MIT, I had just finished an introductory course in classical mechanics that so impressed me that to this day I cannot watch a plane in flight—especially in a turn—without imaging it bristling with vectors. Near the end of the course the professor showed that, if an airplane is treated as a rigid body, there arises a mysterious collection of rather simple-looking integrals called the components of the moment of inertia tensor. Tensor—what power those two syllables seemed to resonate. I had heard the word once before, in an aside by a graduate instructor to the *cognoscenti* in the front row of a course in strength of materials. "What the book calls stress is actually a tensor. . . ."

With my interest twice piqued and with time off from fighting the brushfires of a demanding curriculum, I was ready for my first serious effort at self-instruction. In Los Angeles, after several tries, I found a store with a book on tensor analysis. In my mind I had rehearsed the scene in which a graduate student or professor, spying me there, would shout, "You're an undergraduate. What are you doing looking at a book on tensors?" But luck was mine: the book had a plain brown dust jacket. Alone in my room, I turned immediately to the definition of a tensor: "A 2nd order tensor is a collection of n^2 objects that transform according to the rule . . ." and thence followed an inscrutable collection of superscripts, subscripts, overbars, and partial derivatives. A pedagogical disaster! Where was the connection with those beautiful, simple, boldfaced symbols, those arrows that I could visualize so well?

I was not to find out until after graduate school. But it is my hope that, with this book, you, as an undergraduate, may sail beyond that bar on which I once floundered. You will find that I take nearly three chapters to prepare you for the shock of the tensor transformation formulas. I don't try to hide them—they're the only equations in the book that are boxed. But long before, about halfway through Chapter 1, I tell you what a 2nd order tensor *really* is—a linear operator that sends vectors into vectors. If you apply the stress tensor to the unit normal to a plane through a point in a body, then out comes the stress vector, the force/area acting across the plane at that point. (That the stress vector is linear in the unit normal, i.e., that a stress tensor even exists, is a gift of nature; nonlinearity is more often the rule.) The subsequent "débauche des indices" that follows this tidy definition of a 2nd order tensor is the result of exposing the gears of a machine for grinding out the workings of a tensor. Abolish the machine and there is no hope of producing numerical results except in the simplest of cases.

This book falls into halves: Algebra and Calculus. The first half of the first half (Chapter 1) emphasizes concepts. Here, I have made a special effort to relate the mathematical and physical notions of a vector. I acknowledge my debt to Hoffman's intriguing little book, *About Vectors* (Dover, 1975). (But there are points where we differ—I disagree with his contention that vectors cannot represent finite rotations.) Chapter 2 deals mostly with the index apparatus necessary to represent and manipulate vectors and tensors in general bases. Chapter 3, through the vehicle of Newton's law of motion, introduces moving frames and the Christoffel symbols. To help keep the basic kinematic ideas and their tensor generalizations in mind simultaneously, I list a number of equations in dual form, a device that I have found successful in the classroom. The last chapter starts with a homely example of the gradient and builds to the covariant derivative. Throughout this chapter there are applications to continuum mechanics. Although the basic equations (excluding electricity and magnetism) were known by the 1850's, it was only under the spur of general relativity that tensor analysis began to diffuse into this older field. (In my own specialty, shell theory, tensor analysis did not appear until the early 1940's, in the Soviet literature, even though the underlying theory of surfaces and their tensor description had been central to the understanding of general relativity.)

I have provided no systematic lists of grad, div, curl, etc. in various coordinate systems. Such useful information can be found in Magnus, Oberhettinger, and Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics,* 3rd enlarged edition, Chapter XII, Springer-Verlag 1966; or in Gradshteyn and Ryzhik, *Tables of Integrals, Series and Products,* 4th edition, corrected and enlarged, Academic Press, 1980.

It is a happy thought that much of the drudgery involved in expanding equations and verifying solutions in specific coordinate systems can now be done by computers, programmed to do symbol manipulation. The interested reader should consult "Computer Symbolic Math in Physics Eduation," by D. R. Stoutemyer, *Am. J. Phys.*, vol. 49 (1981), pp. 85–88, or "A Review of Algebraic Computing in General Relativity," by R. A. d'Inverno, Chapter 16 of *General Relativity and Gravitation,* vol. 1, ed. A. Held, Plenum Press, N.Y. and London, 1980.

I am pleased to acknowledge the help of three friends: Mark Duva, a former student, who, in his gracious but profound way, let me get away with nothing in class; Bruce Chartres, who let me filter much of this book through his fine mind; and Ernst Soudek, who, though not a native speaker, tuned the final manuscript with his keen ear for English.

Finally, my thanks to Carolyn Duprey and Ruth Nissley, who typed the original manuscript, and then with patience and good humor, retyped what must have seemed to be hundreds of petty changes.

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