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Matrix Groups

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To my teacher and friend
Raymond Louis Wilder
this book is affectionately dedicated.

Introduction

These notes were developed from a course taught at Rice University in the spring of 1976 and again at the University of Hawaii in the spring of 1977. It is assumed that the students know some linear algebra and a little about differentiation of vector-valued functions. The idea is to introduce students to some of the concepts of Lie group theory-- all done at the concrete level of matrix groups. As much as we could, we motivated developments as a means of deciding when two matrix groups (with different definitions) are isomorphic.

In Chapter I "group" is defined and examples are given; homomorphism and isomorphism are defined. For a field k , $M_n(k)$ denotes the algebra of $n \times n$ matrices over k . We recall that $A \in M_n(k)$ has an inverse if and only if $\det A \neq 0$, and define the general linear group $GL(n,k)$. We construct the skew-field $\mathbb{H}$ of quaternions and note that for $A \in M_n(\mathbb{H})$ to operate linearly on $\mathbb{H}^n$ we must operate on the right (since we multiply a vector by a scalar on the left). So we use row vectors for $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{H}^n$ and write xA for the row vector obtained by matrix multiplication. We get a <u>complex-valued</u> determinant function on $M_n(\mathbb{H})$ such that $\det A \neq 0$ guarantees that A has an inverse.

Chapter II introduces conjugation on $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$ and then an inner product $\langle \, , \, \rangle$. Basic properties of $\langle \, , \, \rangle$ are given and then for $k \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$ we define the orthogonal group

$$\mathbb{O}(n,k) = \{A \in M_n(k) \mid \langle xA,yA\rangle = \langle x,y\rangle \text{ for all } x,y \in k^n\} .$$

$\mathcal{O}(n,\mathbb{R})$ is written $\mathcal{O}(n)$ and called the <u>orthogonal</u> group. $\mathcal{O}(n,\mathbb{C})$ is written $U(n)$ and called the <u>unitary</u> group. $\mathcal{O}(n,\mathbb{H})$ is written $Sp(n)$ and called the <u>symplectic</u> group. If $A \in \mathcal{O}(n)$ then $\det A \in \{1,-1\}$ and the subgroup with $\det = 1$ is denoted by $SO(n)$ and called the <u>special orthogonal group</u>. If $A \in U(n)$ then $\det A$ is a complex number of unit length. The subgroup with $\det = 1$ is denoted by $SU(n)$ and called the <u>special unitary group</u>. As a first example of a matrix group isomorphism we show that $Sp(1) \cong SU(2)$.

In Chapter III we define the first invariant (i.e., something unchanged by an isomorphism) of a matrix group; namely, its <u>dimension</u>. A tangent vector to a matrix group G is $\gamma'(0)$ for some differentiable curve γ in G with $\gamma(0) = I$. The set T_G of all tangent vectors is shown to be a vector space, a real subspace of $M_n(k)$ ($k \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$). The dimension of T_G (as a real vector space) is the dimension of G . Smooth homomorphisms are defined and shown to induce linear maps of tangent spaces. Then dimension is seen to be an invariant.

In order, in Chapter IV, to calculate the dimensions of our matrix groups we develop the exponential map $\exp\colon M_n(k) \to GL(n,k)$ and the logarithm, $\log\colon U \to M_n(k)$ where U in some neighborhood of I in $GL(n,k)$. We have that exp and log are inverses, $\exp\colon V \to U$ and $\log\colon U \to V$ where V is a neighborhood of 0 on $M_n(k)$ and U is a neighborhood of I in $M_n(k)$ (actually $U \subset GL(n,k)$). One-parameter subgroups are defined and proved to be determined by their derivatives at 0 . It follows that T_G can be taken to be all derivatives of one-parameter subgroups. Lie algebras are defined and we see that each T_G is a Lie algebra. Finally, we then calculate the dimensions of SO(n) , U(n) , SU(n) and Sp(n) .

In Chapter V we consider the very specific question of whether Sp(1) and SO(3) are isomorphic. We get a surjective homomorphism ρ: Sp(1) → SO(3) with kernel = {1,-1} . Then we define the center of a group, show it is an invariant and then calculate Center Sp(1) = {1,-1} and Center SO(3) = {I} , proving that Sp(1) $\not\cong$ SO(3) . We define quotient groups and then note that we get new groups $\frac{G}{\text{center}}$ whenever a matrix group has nontrivial center.

In Chapter VI we do some topology which is needed in other parts of the text. All of our matrix groups are in some euclidean space and we just do topology of subsets of euclidean spaces. We give some basic results about continuity of functions, connected sets and compact sets. The proof that continuous functions preserve compactness relegated to an appendix. We consider countable bases for open sets since this is needed later in our study of maximal tori in matrix groups. Finally, there is a short section on manifolds.

Chapters VII, VIII, and IX are devoted to studying maximal tori in our matrix groups. We describe certain specific maximal tori. We prove that any two maximal tori are conjugate and that, if G is connected, then these conjugates cover G . At this stage we then know the dimension, center and rank of all of our matrix groups and these suffice to settle our original question as to which of these groups are isomorphic. At the end of Chapter IX we discuss simple groups and covering groups. The only new groups which arise are the double covers of SO(n) (n = 3,4,...). This leads to the question:

Is the double cover of SO(2n+1) isomorphic with Sp(n) ?

In Chapter X we construct the double cover Spin(n) of SO(n) using Clifford algebras. We show that $\mathrm{Spin}(1) \cong S^0$, $\mathrm{Spin}(2) \cong S^1$, $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$ $(= S^3)$ and $\mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. Finally we show that

$$\mathrm{Spin}(5) \cong \mathrm{Sp}(2) \quad \text{and} \quad \mathrm{Spin}(6) \cong \mathrm{SU}(4) .$$

In Chapter XI we finish our job by showing that $\mathrm{Sp}(n) \not\cong \mathrm{Spin}(2n+1)$ for any $n > 2$. This is done by looking at normalizers of maximal tori and resulting Weyl groups. If the normalizer is the semidirect product of the torus and the Weyl group, we say that the normalizer splits. If Spin(2n+1) and Sp(n) were isomorphic we would have

$$\frac{\mathrm{Spin}(2n+1)}{\text{center}} \cong \frac{\mathrm{Sp}(n)}{\text{center}} .$$

So our result is a consequence of the following three results.

(*) The normalizer in Sp(n) does not split for any n .

(†) The normalizer in $\frac{\mathrm{Sp}(n)}{\text{center}}$ splits $\Leftrightarrow n \in \{1,2\}$.

(Υ) The normalizer in $\frac{\mathrm{Spin}(2n+1)}{\text{center}}$ $(= \mathrm{SO}(2n+1))$ splits for $n = 1,2,3,\ldots$.

Finally, in Chapter XII we give a brief introduction to abstract Lie groups.

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