

# Developments in Mathematics

---

VOLUME 34

---

*Series Editors:*

Krishnaswami Alladi, *University of Florida, Gainesville, FL, USA*

Hershel M. Farkas, *Hebrew University of Jerusalem, Jerusalem, Israel*

For further volumes:

[www.springer.com/series/5834](http://www.springer.com/series/5834)

Simeon Reich • Alexander J. Zaslavski

# Genericity in Nonlinear Analysis

 Springer

Simeon Reich  
Department of Mathematics  
Technion-Israel Institute of Technology  
Haifa, Israel

Alexander J. Zaslavski  
Department of Mathematics  
Technion-Israel Institute of Technology  
Haifa, Israel

ISSN 1389-2177

Developments in Mathematics

ISBN 978-1-4614-9532-1

DOI 10.1007/978-1-4614-9533-8

Springer New York Heidelberg Dordrecht London

ISSN 2197-795X (electronic)

ISBN 978-1-4614-9533-8 (eBook)

Library of Congress Control Number: 2013955299

Mathematics Subject Classification: 06-xx, 34-xx, 46-xx, 47-xx, 49-xx, 54-xx, 65-xx, 90-xx

© Springer Science+Business Media New York 2014

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Printed on acid-free paper

Springer is part of Springer Science+Business Media ([www.springer.com](http://www.springer.com))

# Preface

In recent years it has become more and more evident that Nonlinear Functional Analysis is of crucial importance in the Mathematical Sciences. This is because functional analytic ideas and methods have turned out to be essential tools in the analysis of nonlinear phenomena in many areas of Mathematics and its applications. Among these areas one can mention Ordinary Differential Equations, Partial Differential Equations, the Geometry of Banach Spaces, Nonlinear Operator Theory, the Calculus of Variations, Optimal Control Theory, Optimization and Mathematical Economics.

One of the main features of the functional analytic approach is the investigation and solution of general classes of problems rather than of more specific individual ones. When one uses this approach, the following question arises:

We consider a class of problems which is identified with some functional space equipped with a natural complete metric. We know that for some elements of the functional space the corresponding problems possess a solution (or a solution with some desirable properties) and for some elements such solutions do not exist. We usually know some sufficient conditions for the existence of solutions, but often these conditions are difficult to verify or they hold for rather small subsets of the whole space. In such situations it is natural to ask if a solution (or a solution with some desirable properties) exists for most elements of the functional space in the sense of Baire category. This means that the functional space under consideration contains an everywhere dense  $G_\delta$  subset such that for all its elements a solution exists.

It turns out that this generic approach is very useful and many interesting and important problems can be solved using it. The goal of our book is to demonstrate this. Although it is, of course, impossible to cover the whole spectrum of present-day trends in Nonlinear Analysis and its applications where the generic approach is used, we do present quite a few of the main topics which are of current research interest. They include fixed point theory of both single- and set-valued mappings, convergence analysis of infinite products, best approximation problems, discrete and continuous descent methods for minimization in a general Banach space, and

the structure of minimal energy configurations with rational numbers in the Aubry-Mather theory.

Now we describe the structure of the book. We begin in Chap. 1 with the applications of the Baire theory to fixed point theory. A self-mapping of a complete metric space is called nonexpansive if it is Lipschitz with Lipschitz constant one. If the Lipschitz constant is less than one, then it is called a strict contraction. According to Banach's celebrated result, a strict contraction has a unique fixed point and all its iterates converge to it. It was unclear what happens when a mapping acting on a closed and convex subset of a general Banach space is just nonexpansive until the classical paper by De Blasi and Myjak of 1976 [49], where they show, using the Baire approach, that most mappings in the class of nonexpansive self-mappings of a bounded, closed and convex subset of a general Banach space possess a unique fixed point which attracts uniformly all their iterates. Note that they also show that the subclass of strict contractions is a small set in the whole class of nonexpansive mappings.

Chapter 2 is devoted to further generalizations, extensions and developments concerning this result of De Blasi and Myjak. Using the Baire approach, we establish existence and uniqueness of a fixed point for a generic mapping, convergence of iterates of a generic nonexpansive mapping, stability of the fixed point under small perturbations of a mapping, convergence of Krasnosel'skii-Mann iterations of nonexpansive mappings, generic power convergence of order preserving mappings, and existence and uniqueness of positive eigenvalues and eigenvectors of order-preserving linear operators. In this chapter we also study convergence of iterates of nonexpansive mappings in the presence of computational errors.

Chapter 3 is devoted to an important subclass of the class of nonexpansive mappings which consists of the so-called contractive mappings. A contractive mapping is obtained if in the definition of a strict contraction the constant is replaced by a monotonically decreasing function with nonnegative values which do not exceed one and which is a function of the distance between two points. This topic has recently become rather popular. In Chap. 3 we study different types of contractive mappings, existence of fixed points for such mappings, convergence of their powers to a fixed point, stability of a fixed point under small perturbations of the mapping, and use the Baire approach to show that most nonexpansive mappings are contractive.

In Chap. 4 we use the generic approach in order to study the asymptotic behavior of trajectories of a certain dynamical system which originates in a convex minimization problem. Usually, an algorithm for the minimization of an objective function on a set can be considered a self-mapping of the set for which the objective function is a Lyapunov function. In our case the set is a closed subset of a Banach space. The results presented in this chapter show that for most algorithms, the values of the objective function along all the trajectories tend to its infimum.

In Chap. 5 we generalize some of the results of Chap. 2 for mappings which are relatively nonexpansive with respect to Bregman distances. Such mappings appear in optimization theory and in studies of feasibility problems [37, 39].

Chapter 6 is devoted to the study of convergence of infinite products of different classes of mappings. The convergence of infinite products of nonexpansive mappings is of major importance because of their many applications in the study of feasibility and optimization problems. We study the convergence of typical (generic) infinite products of mappings to the set of their common fixed points, and establish weak ergodic theorems (a term which originates in population biology), which roughly mean that all trajectories generated by infinite products converge to each other. We study convergence and its stability for generic infinite products of nonexpansive mappings, uniformly continuous mappings, order-preserving mappings, order-preserving linear mappings, homogeneous order-preserving mappings, products of affine mappings, as well as products of resolvents of accretive operators.

In Chap. 7 we study best approximation problems in a general Banach space. A best approximation problem is determined by a pair consisting of a point and a closed (convex) subset of a Banach space. We consider the complete metric space of such pairs equipped with a natural complete metric and show that for most (in the sense of Baire category) pairs the corresponding best approximation problem has a unique solution. We also provide some generalizations and extensions of this result.

In Chap. 8 we study discrete and continuous descent methods for minimizing a convex (Lipschitz) function on a general Banach space. We consider a space of vector fields  $V$  such that for any point  $x$  in the Banach space, the directional derivative in the direction  $Vx$  is nonpositive. This space of vector fields is equipped with a complete metric. Each vector field generates two gradient type algorithms (discrete descent methods) and a flow which consists of the solutions of the corresponding evolution equation (continuous descent method). We show that most (in the sense of Baire category) vector fields produce algorithms for which values of the objective function tend to its infimum as  $t$  tends to infinity. Actually, we introduce the subclass of regular vector fields, show that the convergence property stated above holds for them and that a generic vector field is regular. We also show that this convergence property is stable under small perturbations of a given regular vector field.

Chapter 9 is devoted to set-valued mappings. We study approximate fixed points of such mappings, existence of fixed points, and the convergence and stability of iterates of set-valued mappings.

Chapter 10 is devoted to the Aubry-Mather theory applied to the famous Frenkel-Kontorova model, an infinite discrete model of solid-state physics related to dislocations in one-dimensional crystals. In this model a configuration of a system is a sequence of real numbers with indices from  $-\infty$  to  $+\infty$ . We are interested in ( $h$ )-minimal configurations with respect to an energy function  $h$ . A configuration is called ( $h$ )-minimal if its total energy cannot be made less by changing its final states. Classical Aubry-Mather theory is concerned with finding and investigating  $h$ -minimal configurations with a given rotation number, where the function  $h$  is fixed. It implies that the set of all periodic  $h$ -minimal configurations of a rational rotation number  $p/q$  is totally ordered. Moreover, between any two neighboring periodic  $h$ -minimal configurations with rotation number  $p/q$ , there are (non-periodic)  $h$ -minimal heteroclinic connections having the same rotation number  $p/q$ . We consider a complete metric space of energy functions  $h$  equipped with a certain  $C^2$

topology and show that for most energy functions in this space, there exist three different  $h$ -minimal configurations with rotation number  $p/q$  such that any other  $h$ -minimal configuration with the same rotation number  $p/q$  is a translation of one of these three.

Haifa  
December 31, 2012

Simeon Reich  
Alexander J. Zaslavski

# Contents

<b>1</b>	<b>Introduction</b>	1
1.1	Hyperbolic Spaces	1
1.2	Successive Approximations	2
1.3	Contractive Mappings	3
1.4	Infinite Products	5
1.5	Contractive Set-Valued Mappings	7
1.6	Nonexpansive Set-Valued Mappings	9
1.7	Porosity	10
1.8	Examples	12
<b>2</b>	<b>Fixed Point Results and Convergence of Powers of Operators</b>	15
2.1	Convergence of Iterates for a Class of Nonlinear Mappings	15
2.2	Convergence of Iterates of Typical Nonexpansive Mappings	23
2.3	A Stability Result in Fixed Point Theory	29
2.4	Well-Posed Null and Fixed Point Problems	34
2.5	Mappings in a Finite-Dimensional Euclidean Space	37
2.6	Approximate Fixed Points	42
2.7	Generic Existence of Small Invariant Sets	47
2.8	Many Nonexpansive Mappings Are Strict Contractions	51
2.9	Krasnosel'skii-Mann Iterations of Nonexpansive Operators	55
2.10	Power Convergence of Order-Preserving Mappings	63
2.11	Positive Eigenvalues and Eigenvectors	72
2.12	Proof of Theorem 2.48	75
2.13	Auxiliary Results for Theorems 2.49–2.51	78
2.14	Proofs of Theorems 2.49 and 2.50	83
2.15	Proof of Theorem 2.51	85
2.16	Convergence of Inexact Orbits for a Class of Operators	87
2.17	Proofs of Theorem 2.65 and Corollary 2.66	89
2.18	Proof of Theorem 2.67	92
2.19	Proof of Theorem 2.68	93
2.20	Proof of Theorem 2.69	96



2.21	Inexact Orbits of Nonexpansive Operators . . . . .	97
2.22	Convergence to Attracting Sets . . . . .	100
2.23	Nonconvergence to Attracting Sets . . . . .	103
2.24	Convergence and Nonconvergence to Fixed Points . . . . .	106
2.25	Convergence to Compact Sets . . . . .	110
2.26	An Example of Nonconvergence to Compact Sets . . . . .	113
<b>3</b>	<b>Contractive Mappings . . . . .</b>	<b>119</b>
3.1	Many Nonexpansive Mappings Are Contractive . . . . .	119
3.2	Attractive Sets . . . . .	121
3.3	Attractive Subsets of Unbounded Spaces . . . . .	124
3.4	A Contractive Mapping with no Strictly Contractive Powers . . . . .	129
3.5	A Power Convergent Mapping with no Contractive Powers . . . . .	132
3.6	A Mapping with Nonuniformly Convergent Powers . . . . .	134
3.7	Two Results in Metric Fixed Point Theory . . . . .	136
3.8	A Result on Rakotch Contractions . . . . .	144
3.9	Asymptotic Contractions . . . . .	149
3.10	Uniform Convergence of Iterates . . . . .	153
3.11	Well-Posedness of Fixed Point Problems . . . . .	157
3.12	A Class of Mappings of Contractive Type . . . . .	159
3.13	A Fixed Point Theorem for Matkowski Contractions . . . . .	166
3.14	Jachymski-Schröder-Stein Contractions . . . . .	170
3.15	Two Results on Jachymski-Schröder-Stein Contractions . . . . .	175
<b>4</b>	<b>Dynamical Systems with Convex Lyapunov Functions . . . . .</b>	<b>181</b>
4.1	Minimization of Convex Functionals . . . . .	181
4.2	Proof of Proposition 4.3 . . . . .	183
4.3	Proofs of Theorems 4.1 and 4.2 . . . . .	185
4.4	Examples . . . . .	188
4.5	Normal Mappings . . . . .	190
4.6	Existence of a Normal $A \in \mathcal{A}_c$ . . . . .	192
4.7	Auxiliary Results . . . . .	193
4.8	Proof of Theorem 4.12 . . . . .	194
4.9	Proof of Theorem 4.13 . . . . .	195
4.10	Proof of Theorem 4.14 . . . . .	196
4.11	Normality and Porosity . . . . .	197
4.12	Proof of Theorem 4.18 . . . . .	198
4.13	Proof of Theorem 4.19 . . . . .	200
4.14	Convex Functions Possessing a Sharp Minimum . . . . .	202
<b>5</b>	<b>Relatively Nonexpansive Operators with Respect to Bregman Distances . . . . .</b>	<b>205</b>
5.1	Power Convergence of Operators in Banach Spaces . . . . .	205
5.2	Power Convergence for a Class of Continuous Mappings . . . . .	206
5.3	Preliminary Lemmata for Theorems 5.1–5.6 . . . . .	209
5.4	Proofs of Theorems 5.1–5.6 . . . . .	213

5.5	A Class of Uniformly Continuous Mappings . . . . .	217
5.6	An Auxiliary Result . . . . .	218
5.7	Proofs of Theorems 5.11 and 5.12 . . . . .	219
5.8	Mappings with a Uniformly Continuous Bregman Function . . . . .	222
5.9	Proofs of Theorems 5.15 and 5.16 . . . . .	223
5.10	Generic Power Convergence to a Retraction . . . . .	226
5.11	Two Lemmata . . . . .	228
5.12	Convergence of Powers of Uniformly Continuous Mappings . . . . .	230
5.13	Convergence to a Retraction . . . . .	231
5.14	Auxiliary Results . . . . .	231
5.15	Proof of Theorem 5.21 . . . . .	233
5.16	Proofs of Theorems 5.22 and 5.23 . . . . .	235
5.17	Convergence of Powers for a Class of Continuous Operators . . . . .	241
5.18	Proofs of Theorems 5.32–5.34 . . . . .	242
<b>6</b>	<b>Infinite Products . . . . .</b>	<b>247</b>
6.1	Nonexpansive and Uniformly Continuous Operators . . . . .	247
6.2	Asymptotic Behavior . . . . .	249
6.3	Nonexpansive Retractions . . . . .	250
6.4	Preliminary Results . . . . .	252
6.5	Proofs of Theorems 6.1 and 6.2 . . . . .	255
6.6	Proofs of Theorems 6.3 and 6.4 . . . . .	257
6.7	Proofs of Theorems 6.5, 6.6 and 6.7 . . . . .	259
6.8	Hyperbolic Spaces . . . . .	263
6.9	Infinite Products of Order-Preserving Mappings . . . . .	263
6.10	Existence of a Unique Fixed Point . . . . .	265
6.11	Asymptotic Behavior . . . . .	269
6.12	Preliminary Lemmata for Theorems 6.16–6.20 . . . . .	271
6.13	Proofs of Theorems 6.16 and 6.17 . . . . .	276
6.14	Proofs of Theorems 6.18 and 6.19 . . . . .	277
6.15	Proof of Theorem 6.20 . . . . .	280
6.16	Infinite Products of Positive Linear Operators . . . . .	282
6.17	Proof of Theorem 6.24 . . . . .	287
6.18	Proof of Theorem 6.26 . . . . .	290
6.19	Proof of Theorem 6.27 . . . . .	295
6.20	Homogeneous Order-Preserving Mappings . . . . .	301
6.21	Preliminary Lemmata for Theorems 6.41–6.43 . . . . .	305
6.22	Proofs of Theorems 6.41 and 6.42 . . . . .	314
6.23	Proof of Theorem 6.43 . . . . .	315
6.24	Infinite Products of Affine Operators . . . . .	321
6.25	A Generic Fixed Point Theorem for Affine Mappings . . . . .	323
6.26	A Weak Ergodic Theorem for Affine Mappings . . . . .	327
6.27	Affine Mappings with a Common Fixed Point . . . . .	329
6.28	Proofs of Theorems 6.64, 6.65 and 6.66 . . . . .	330
6.29	Weak Convergence . . . . .	334
6.30	Proofs of Theorems 6.67 and 6.68 . . . . .	335

6.31 Affine Mappings with a Common Set of Fixed Points . . . . . 336

6.32 Infinite Products of Resolvents of Accretive Operators . . . . . 339

6.33 Auxiliary Results . . . . . 343

6.34 Proof of Theorem 6.71 . . . . . 345

6.35 Proof of Theorem 6.72 . . . . . 348

**7 Best Approximation . . . . . 353**

7.1 Well-Posedness and Porosity . . . . . 353

7.2 Auxiliary Results . . . . . 357

7.3 Proofs of Theorems 7.3–7.5 . . . . . 360

7.4 Generalized Best Approximation Problems . . . . . 363

7.5 Theorems 7.8–7.11 . . . . . 365

7.6 A Basic Lemma . . . . . 367

7.7 Proofs of Theorems 7.8–7.11 . . . . . 372

7.8 A Porosity Result in Best Approximation Theory . . . . . 374

7.9 Two Lemmata . . . . . 375

7.10 Proof of Theorem 7.13 . . . . . 379

7.11 Porous Sets and Generalized Best Approximation Problems . . . 380

7.12 A Basic Lemma . . . . . 383

7.13 Proofs of Theorems 7.16–7.18 . . . . . 389

**8 Descent Methods . . . . . 397**

8.1 Discrete Descent Methods for a Convex Objective Function . . . 397

8.2 An Auxiliary Result . . . . . 401

8.3 Proof of Theorem 8.2 . . . . . 403

8.4 A Basic Lemma . . . . . 406

8.5 Proofs of Theorems 8.3 and 8.4 . . . . . 409

8.6 Methods for a Nonconvex Objective Function . . . . . 412

8.7 An Auxiliary Result . . . . . 416

8.8 Proof of Theorem 8.8 . . . . . 417

8.9 A Basic Lemma for Theorems 8.9 and 8.10 . . . . . 419

8.10 Proofs of Theorems 8.9 and 8.10 . . . . . 421

8.11 Continuous Descent Methods . . . . . 424

8.12 Proof of Theorem 8.14 . . . . . 427

8.13 Proof of Theorem 8.15 . . . . . 428

8.14 Regular Vector-Fields . . . . . 432

8.15 Proofs of Propositions 8.16 and 8.17 . . . . . 434

8.16 An Auxiliary Result . . . . . 436

8.17 Proof of Theorem 8.18 . . . . . 437

8.18 Proof of Theorem 8.19 . . . . . 438

8.19 Proof of Theorem 8.20 . . . . . 439

8.20 Proof of Theorem 8.21 . . . . . 440

8.21 Most Continuous Descent Methods Converge . . . . . 441

8.22 Proof of Proposition 8.24 . . . . . 442

8.23 Proof of Theorem 8.25 . . . . . 443

**9 Set-Valued Mappings** . . . . . 449

9.1 Contractive Mappings . . . . . 449

9.2 Star-Shaped Spaces . . . . . 451

9.3 Convergence of Iterates of Set-Valued Mappings . . . . . 453

9.4 Existence of Fixed Points . . . . . 457

9.5 An Auxiliary Result and the Proof of Proposition 9.16 . . . . . 459

9.6 Proof of Theorem 9.14 . . . . . 460

9.7 Proof of Theorem 9.15 . . . . . 461

9.8 An Extension of Theorem 9.15 . . . . . 462

9.9 Generic Existence of Fixed Points . . . . . 464

9.10 Topological Structure of the Fixed Point Set . . . . . 470

9.11 Approximation of Fixed Points . . . . . 473

9.12 Approximating Fixed Points in Caristi’s Theorem . . . . . 479

**10 Minimal Configurations in the Aubry-Mather Theory** . . . . . 481

10.1 Preliminaries . . . . . 481

10.2 Spaces of Functions . . . . . 484

10.3 The Main Results . . . . . 486

10.4 Preliminary Results for Assertion 1 of Theorem 10.9 . . . . . 487

10.5 Preliminary Results for Assertion 2 of Theorem 10.9 . . . . . 493

10.6 Proof of Proposition 10.11 . . . . . 502

**References** . . . . . 513

**Index** . . . . . 519