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Genericity in Nonlinear Analysis

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Preface

In recent years it has become more and more evident that Nonlinear Functional Analysis is of crucial importance in the Mathematical Sciences. This is because functional analytic ideas and methods have turned out to be essential tools in the analysis of nonlinear phenomena in many areas of Mathematics and its applications. Among these areas one can mention Ordinary Differential Equations, Partial Differential Equations, the Geometry of Banach Spaces, Nonlinear Operator Theory, the Calculus of Variations, Optimal Control Theory, Optimization and Mathematical Economics.

One of the main features of the functional analytic approach is the investigation and solution of general classes of problems rather than of more specific individual ones. When one uses this approach, the following question arises:

We consider a class of problems which is identified with some functional space equipped with a natural complete metric. We know that for some elements of the functional space the corresponding problems possess a solution (or a solution with some desirable properties) and for some elements such solutions do not exist. We usually know some sufficient conditions for the existence of solutions, but often these conditions are difficult to verify or they hold for rather small subsets of the whole space. In such situations it is natural to ask if a solution (or a solution with some desirable properties) exists for most elements of the functional space in the sense of Baire category. This means that the functional space under consideration contains an everywhere dense G_δ subset such that for all its elements a solution exists.

It turns out that this generic approach is very useful and many interesting and important problems can be solved using it. The goal of our book is to demonstrate this. Although it is, of course, impossible to cover the whole spectrum of present-day trends in Nonlinear Analysis and its applications where the generic approach is used, we do present quite a few of the main topics which are of current research interest. They include fixed point theory of both single- and set-valued mappings, convergence analysis of infinite products, best approximation problems, discrete and continuous descent methods for minimization in a general Banach space, and the structure of minimal energy configurations with rational numbers in the Aubry-Mather theory.

Now we describe the structure of the book. We begin in Chap. 1 with the applications of the Baire theory to fixed point theory. A self-mapping of a complete metric space is called nonexpansive if it is Lipschitz with Lipschitz constant one. If the Lipschitz constant is less than one, then it is called a strict contraction. According to Banach's celebrated result, a strict contraction has a unique fixed point and all its iterates converge to it. It was unclear what happens when a mapping acting on a closed and convex subset of a general Banach space is just nonexpansive until the classical paper by De Blasi and Myjak of 1976 [49], where they show, using the Baire approach, that most mappings in the class of nonexpansive self-mappings of a bounded, closed and convex subset of a general Banach space possess a unique fixed point which attracts uniformly all their iterates. Note that they also show that the subclass of strict contractions is a small set in the whole class of nonexpansive mappings.

Chapter 2 is devoted to further generalizations, extensions and developments concerning this result of De Blasi and Myjak. Using the Baire approach, we establish existence and uniqueness of a fixed point for a generic mapping, convergence of iterates of a generic nonexpansive mapping, stability of the fixed point under small perturbations of a mapping, convergence of Krasnosel'skii-Mann iterations of nonexpansive mappings, generic power convergence of order preserving mappings, and existence and uniqueness of positive eigenvalues and eigenvectors of order-preserving linear operators. In this chapter we also study convergence of iterates of nonexpansive mappings in the presence of computational errors.

Chapter 3 is devoted to an important subclass of the class of nonexpansive mappings which consists of the so-called contractive mappings. A contractive mapping is obtained if in the definition of a strict contraction the constant is replaced by a monotonically decreasing function with nonnegative values which do not exceed one and which is a function of the distance between two points. This topic has recently become rather popular. In Chap. 3 we study different types of contractive mappings, existence of fixed points for such mappings, convergence of their powers to a fixed point, stability of a fixed point under small perturbations of the mapping, and use the Baire approach to show that most nonexpansive mappings are contractive.

In Chap. 4 we use the generic approach in order to study the asymptotic behavior of trajectories of a certain dynamical system which originates in a convex minimization problem. Usually, an algorithm for the minimization of an objective function on a set can be considered a self-mapping of the set for which the objective function is a Lyapunov function. In our case the set is a closed subset of a Banach space. The results presented in this chapter show that for most algorithms, the values of the objective function along all the trajectories tend to its infimum.

In Chap. 5 we generalize some of the results of Chap. 2 for mappings which are relatively nonexpansive with respect to Bregman distances. Such mappings appear in optimization theory and in studies of feasibility problems [37, 39].

Chapter 6 is devoted to the study of convergence of infinite products of different classes of mappings. The convergence of infinite products of nonexpansive mappings is of major importance because of their many applications in the study of feasibility and optimization problems. We study the convergence of typical (generic) infinite products of mappings to the set of their common fixed points, and establish weak ergodic theorems (a term which originates in population biology), which roughly mean that all trajectories generated by infinite products converge to each other. We study convergence and its stability for generic infinite products of nonexpansive mappings, uniformly continuous mappings, order-preserving mappings, order-preserving linear mappings, homogeneous order-preserving mappings, products of affine mappings, as well as products of resolvents of accretive operators.

In Chap. 7 we study best approximation problems in a general Banach space. A best approximation problem is determined by a pair consisting of a point and a closed (convex) subset of a Banach space. We consider the complete metric space of such pairs equipped with a natural complete metric and show that for most (in the sense of Baire category) pairs the corresponding best approximation problem has a unique solution. We also provide some generalizations and extensions of this result.

In Chap. 8 we study discrete and continuous descent methods for minimizing a convex (Lipschitz) function on a general Banach space. We consider a space of vector fields V such that for any point x in the Banach space, the directional derivative in the direction Vx is nonpositive. This space of vector fields is equipped with a complete metric. Each vector field generates two gradient type algorithms (discrete descent methods) and a flow which consists of the solutions of the corresponding evolution equation (continuous descent method). We show that most (in the sense of Baire category) vector fields produce algorithms for which values of the objective function tend to its infimum as t tends to infinity. Actually, we introduce the subclass of regular vector fields, show that the convergence property stated above holds for them and that a generic vector field is regular. We also show that this convergence property is stable under small perturbations of a given regular vector field.

Chapter 9 is devoted to set-valued mappings. We study approximate fixed points of such mappings, existence of fixed points, and the convergence and stability of iterates of set-valued mappings.

Chapter 10 is devoted to the Aubry-Mather theory applied to the famous Frenkel-Kontorova model, an infinite discrete model of solid-state physics related to dislocations in one-dimensional crystals. In this model a configuration of a system is a sequence of real numbers with indices from $-\infty$ to $+\infty$. We are interested in (h)-minimal configurations with respect to an energy function h. A configuration is called (h)-minimal if its total energy cannot be made less by changing its final states. Classical Aubry-Mather theory is concerned with finding and investigating h-minimal configurations with a given rotation number, where the function h is fixed. It implies that the set of all periodic h-minimal configurations of a rational rotation number p/q is totally ordered. Moreover, between any two neighboring periodic h-minimal configurations with rotation number p/q, there are (non-periodic) h-minimal heteroclinic connections having the same rotation number p/q. We consider a complete metric space of energy functions h equipped with a certain C^2

topology and show that for most energy functions in this space, there exist three different h-minimal configurations with rotation number p/q such that any other h-minimal configuration with the same rotation number p/q is a translation of one of these three.

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December 31, 2012 Alexander J. Zaslavski

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