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Green's Functions in the Theory of Ordinary Differential Equations

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To my wife Marina and my sons Víctor and Martín

Preface

Ordinary differential equations involving additional conditions on the boundary of a given bounded interval have been exhaustively studied in the literature.

The classical results of Peano and Picard ensure, under suitable assumptions on the regularity of the nonlinear part of the equation, the existence and uniqueness of solutions of the considered initial value problem, which is defined in a neighborhood of the starting point.

It is important to point out that the aforementioned results are related to the concept of "local solution." However, if we are dealing with boundary value problems, the solution must be defined in the whole interval of definition and the local existence has no sense.

To see that the existence of solutions of boundary value problems cannot be deduced exclusively from the regularity data of the equation, it suffices to look for the 1-periodic solutions of the equation $u'(t) = 1$. It is obvious that the general solution of such equation is given by $u_c(t) = c + t$, with $c \in \mathbb{R}$, and none of them satisfies $u(0) = u(1)$. Thus, to ensure the existence of solutions of this kind of problems, we must take into account not only the regularity of the functions that appear in the equation but also the information provided by the boundary conditions.

Under suitable regularity assumptions on the linear operator L, we have that if L is a linear operator and equation $L\,u = f$, coupled with suitable homogeneous linear boundary value conditions on a real interval $[a, b]$, has only the trivial solution for $f \equiv 0$, then the associated linear operator is invertible and its inverse operator, $L^{-1} f$, is characterized by an integral kernel, $g(t, s)$, called Green's function,[1] and the solution of the considered problem is then given by

$$u(t) = L^{-1} f(t) := \int_a^b g(t, s)\, f(s) ds, \quad t \in [a, b].$$

[1]George Green (1793–1841) was the first mathematician to use such kind of kernels to solve boundary value problems.

We notice that, as it has been pointed out in [19], if we are able to obtain the expression of Green's kernel, we know the cases (if the linear operator depends on some parameters, for instance) in which it is not defined and, in consequence, the resonant cases (of nonuniqueness of the homogeneous problem) are explicitly given. The main advantage of Green's function is the fact that it is independent of the function f. To get the exact solution for each particular case of f we only need to calculate the corresponding integral, and so we have the expression that we are looking for (this is due to the fact that Green's function is just the kernel of the operator L^{-1}).

The aim of this monograph is to provide to graduate and doctoral students, together with researchers interested in this field, a comprehensive and thorough study of Green's functions. Along the monograph, some classical results of functional analysis are needed, which, for a better understanding of the text, will be introduced throughout the book as they are used. Along the chapters, some examples are given to illustrate the obtained results. Moreover, some particular cases are introduced to show the necessary conditions that are required to develop the theory.

The properties of these functions are widely used in the literature. But there are few books devoted just to the study of these functions [25, 27, 31, 38, 39, 53, 54]. In many cases, these functions are presented as the only function that verifies certain "a priori" given axioms. Our approach is posed differently; our intention is "to arrive" at Green's function. To do so, we consider, first, in Sect. 1.2, a linear system of first order, for which, using classical results of linear differential equations, we obtain the expression of the kernel of the integral equation that represents the solution we are looking for. This kernel is the so-called Green's function. Moreover, the determinant of a suitable matrix characterizes the uniqueness of such function. This determinant provides us the spectrum of the studied linear operator.

Although the paper is aimed to two-point boundary value problems, a characterization of how to use this theory when we are dealing with multipoint boundary conditions is also shown in Subsect. 1.2.1.

Once the characterization of the existence and uniqueness of Green's functions is presented, we prove some of their basic properties and their relationship to the related linear operator. For example, we will consider in Sect. 1.3 the relationship between the symmetric kernel of Green's function and the linear adjoint operator.

After this, we will focus on the scalar equations of nth order, for which, as in the previous case, we automatically get optimal existence and uniqueness conditions in Sect. 1.4. We also characterize the relationship between the existence and uniqueness of Green's function and the spectrum of the associated nth-order linear operator. Moreover, we will consider the study of the symmetry properties of Green's function and the self-adjoint character of the related linear operator. When the coefficients of the linear operator are constant, different techniques to calculate the exact expression of Green's functions are given. In the particular cases of initial, terminal, and periodic problems, the expression of Green's function follows from the inverse of a related constant matrix.

At this point, we introduce in Sect. 1.5 the method of lower and upper solutions. This tool is very well known in the theory of nonlinear boundary value problems. Only a few particular cases have been considered in this section. We introduce this method here to present different examples that point out the deep influence that the existence and uniqueness of Green's function of a related linear operator has on the existence of solution of nonlinear boundary value problems. More concisely, the existence results for nonlinear problems follow when there is a related Green's function with constant sign. So, it is fundamental to describe the cases in which the linear operators satisfy some suitable comparison principles, i.e., if the linear operator acting over a function has constant sign, then this function must have constant sign too.

These comparison principles are studied in Sect. 1.6. In this case the framework is very general and the concept of related set to a boundary condition is introduced. The equivalence of the validity of a comparison result in a particular related set and the constant sign of a Green's function is proved here. The validity of a comparison principle for a linear operator and for its adjoint is also pointed out.

Next section is devoted to monotone iterative techniques. As in the case of the lower and upper solutions, this is a tool used for nonlinear boundary value problems. In this monograph it is presented in a general framework for nth-order problems. This approach will be fundamental to Sect. 1.8, in which a one parameter family of nth-order linear operators is studied and the monotonicity dependence, with respect to the parameter, of the constant sign Green's functions is proved. Moreover, by using this kind of techniques, we describe the range of the parameters for which the linear operator has constant sign Green's function. This study is closely related to the spectral theory of completely continuous operators.

The last two sections are devoted to present the exact interval of the real parameter for which some particular Green's functions, related to given nth-order linear differential operators, have constant sign on the space of either the periodic or the separated boundary conditions.

The monograph ends with two appendices. The first one is concerned with the algorithm developed in [19] and implemented in a Mathematica program package. This program is of free access and can be downloaded from the web page of the author. It allows the effective calculation of Green's function provided that the differential operator has constant coefficients. These developments allow us to study their properties, both qualitative and quantitative, in a more accessible way. It is important to note that the main difficulty of this type of functions lies not only in its exact calculation but also even in the case where it can be performed, in the complexity of the expression obtained and, therefore, its handling becomes very difficult. Addressing methods of obtaining this expression more easily, we can make the study of its extreme values, its symmetry or some kind of boundedness, with less margin for error.

The second appendix shows the exact expression of a list of the most commonly studied operators. They have been calculated with the package explained previously and they are accessible from the Wolfram web page.

The monograph is completed with a bibliography of papers, both classic and recent, that have contributed to the development of this theory.

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