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Two-Person Zero-Sum Games

Fourth Edition

Springer

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Foreword

Practical men, who believe themselves to be quite exempt from any intellectual influences, are usually the slaves of some defunct economist. . . .

J. M. Keynes

This book is unusual among books on game theory in considering only the special case where there are exactly two players whose interests are completely opposed—the two-person zero-sum (TPZS) case. A few words of explanation are in order about why such an apparently small part of John von Neumann and Oskar Morgenstern's (vN&M's) grander vision in their seminal book *Theory of Games and Economic Behavior* should now itself be thought a suitable subject for a textbook. Our explanation will involve a brief review of the history of game theory and some speculation about its future.

The vN&M book appeared in 1944. It played to rave reviews, not only from academics but in the popular press. There are few instances in history of a theory being born and taken so seriously and so suddenly. The May 1949 issue of *Fortune*, for example, contained a 20-page article describing the theory's accomplishments in World War II and projecting further successes in industry. Williams (1954) wrote *The Compleat Strategyst* as a book intended to make the theory more widely accessible to nonmathematicians. There were strong efforts to further develop the theory, particularly at Princeton University and at the RAND Corporation. The American Mathematical Society published a sequence of four volumes between 1950 and 1964 devoted entirely to the subject. Much of this interest has continued to the present; there are now several dedicated journals, two of which are published by the Game Theory Society.

The initial euphoria was doomed to be disappointed. Game theory has had its practical successes, but applications could hardly have kept pace with initial expectations. An explanation of why this has happened requires us to identify two extreme forms of game. One extreme is single player (SP) games, which can be thought of as games in which all decision makers who can affect the outcome have the same goal. The natural mode of social interaction in SP games is cooperation, since there is no point in competition when all parties have the same goal. Use of the word "game" to describe such situations could even be considered a misuse of the term, but we will persist for the moment. The other extreme is where all players who can affect the outcome have opposite goals. Since the idea of "opposite" gets difficult unless there are exactly two players, this is the TPZS case. The natural mode of social interaction in TPZS games is competition, since neither player has anything to gain by cooperation. Games that fall at neither extreme will be referred to as N-person (NP) games. In general, the behavior of players in NP games can be expected to exhibit aspects of both cooperation and competition. Thus, SP and TPZS games are the extreme cases.

vN&M dealt with NP games, as well as the SP and TPZS specializations. The SP theory had actually been widely applied before the book's appearance, although vN&M made an important contribution to decision making under uncertainty that is reviewed in Chap. 1. One major contribution of the book was to put the theory of finite, TPZS games on a solid basis. The famous "minimax" Theorem asserts the existence of rational behavior as long as randomized strategies are permitted. The vN&M definition of "solution" for TPZS games has stood the test of time, with subsequent years seeing the idea generalized, rather than revised. vN&M also considered rational behavior in NP games, and in fact the greater part of their book is devoted to that topic. However, the NP theory can be distinguished from the TPZS theory in being descriptive rather than normative. It would be good to have a normative theory, since one of our hopes in studying games is to discover "optimal" ways of playing them. There were thus two natural tasks for game theorists in the years following 1944:

1. Generalization and application of the minimax Theorem for TPZS games. The vN&M proof was not constructive, so there were computational issues. Also, the original proof applied only to finite games.
2. Further investigation of what "solution" should mean for NP games. The vN&M concept was not fully understood, and there was the problem of finding a normative concept that could serve as a foundation for applications.

Most effort went initially to the first task, shifting gradually toward the second. In the four volumes published by the American Mathematical Society, the percentage of papers devoted to TPZS games in 1950, 1954, 1957, and 1964 was 52, 47, 30, and 25. This decreasing trend has continued. In 1985, only about 20 % of the papers published in the *International Journal of Game Theory* were devoted to TPZS games, and about 10 % currently (2013). There are several reasons for this shifting emphasis:

1. Theoretical progress in TPZS games came rather quickly. The minimax Theorem or something close to it holds in many infinite games. Given the extended minimax Theorem, there is not much more to be said *in principle* about solutions of TPZS games, even though serious computational and modeling issues remain.
2. While many solution concepts were proposed for NP games, none became widely accepted as a standard. The original vN&M concept was shown to be

defective in 1962, when William Lucas exhibited a game with no solution. This unsettled state of affairs in a theory with such obvious potential for application naturally attracts the attention of theoreticians.

3. Although there are industrial and political applications, TPZS games are applicable mainly to military problems and their surrogates—sports and parlor games. NP games are potentially much more broadly applicable.

With the decline in interest in TPZS games among academics has come a similar decline among practitioners, even among the practicing military analysts who have the most to gain from an awareness of the TPZS theory. The inevitable result is neglect of a relevant, computationally tractable body of theory.

This book was written to make TPZS theory and application accessible to anyone whose quantitative background includes calculus, probability, and enough computer savvy to run Microsoft Excel™. Since the theory has been well worked out by this time, a large part of the book is devoted to the practical aspects of problem formulation and computation. It is intended to be used as the basis for instruction, either self study or formal, and includes numerous exercises.

Preface to the Fourth Edition

This edition of *Two-Person Zero-Sum Games*, the first Springer edition, differs significantly from the previous three INFORMS editions.

- The practice of distributing multiple executable files has been discontinued. Instead, this edition comes with a single Microsoft Excel™ workbook *TPZS.xlsb* that the user is expected to download. The Solver addin that comes with Excel will be sufficient for linear programming exercises.
- Chapters 7–9 are new in this edition, and new material has also been added to the first six chapters.

Please check www.springer.com for the latest version of *TPZS.xlsb*, as well as a list of any errata that have been discovered. If you find any errors in either the book or *TPZS.xlsb*, please describe them to me at the email address below.

My thanks to INFORMS for the publishing support I have received over the last three decades.

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Prerequisites, Conventions, and Notation

This book assumes knowledge of calculus and probability. We will freely refer to such things as the mean and distribution of a random variable, marginal distributions, and other probabilistic concepts. Some knowledge of mathematical programming, especially linear programming, will also be needed because of the intimate relationship between TPZS games and linear programming. Appendix A gives a brief introduction to mathematical programming, and states the notational conventions in use. Appendix B reviews some aspects of convexity, which is also an important topic in game theory.

The TPZS workbook is not absolutely essential if this book is to be used merely as a reference, but readers who are using the book as a text should definitely download it (see the Preface). Many of the exercises are keyed to it. A good practice would be to immediately save it as *TPZS1.xlsb*, thus avoiding modifications to the original, which is not protected in any way. Macros must be enabled.

Chapters are divided into sections and subsections, with subsection a.b.c being subsection c of section b of chapter a. Equation (a.b-n) is the n$^{\text{th}}$ equation of section a.b. Figure a.b is the figure in section a.b, or the a.b-n notation is used when there are more than one. Theorems are numbered similarly. This section-based numbering system has the advantage of quickly determining the section where a figure can be found, but note that the existence of figure 2.2 does not imply the existence of figure 2.1.

Most sums and integrals have an index set that should be well understood from context, so limits are often not shown explicitly. Thus $\sum_i a_i$ is the sum of the numbers a_i over the index set for i. A similar convention is used with the max and min operators. If the index set for i is $\{1,2,3\}$ and if $a_i = i$, then $\sum_i a_i = 6$, $\max_i a_i = 3$, and $\min_i a_i = 1$. The symbol for set inclusion is $\in$, so it is correct to say that $2 \in \{1,2,3\}$. The number of elements in set S is $|S|$, so it is correct to say that $|\{1,4,2\}| = 3$. In the expression $X = \left\{ \mathbf{x} \middle| \mathbf{x} \geq 0 \text{ and } \sum_{i=1}^{m} x_i \leq b \right\}$, the vertical bar is read "such that," so X consists of nonnegative m-vectors such that the sum of the components does not exceed b.

Scalar real numbers or arbitrary set elements are given an italic symbol like x. Boldface, non-italic symbols denote vectors or matrices, so $\mathbf{x} = (x_i)$ is a vector with components x_i. If you see the definition $z \equiv (1,2,3)$, please send me an email complaining about my misuse of my own notation (z should be in bold type). The symbols for a vector and its components will always be identical, except that the components will usually have subscripts. The symbolic notation will not distinguish between matrices and vectors, but matrix examples will be included in brackets [ ] when written out: thus $\begin{bmatrix} -1 & 0 & 3 \\ 2 & 2 & 7.5 \end{bmatrix}$ is a 2×3 matrix.

The end of a theorem is marked with §§§. The statement $a \equiv b$ is simply a definition of a, so a proof is not necessary. The quantity being defined will always be on the left in equalities that use the ≡ symbol.

Random variables, events and sets will always be given upper case, italic symbols. $P(F)$ is the probability of event F and $E(X)$ is the expected value (the mean) of random variable X.

Most chapters include a set of exercises. A partial list of solutions will be found at the end of the book.

The two players involved in a TPZS game are best thought of as humans, so there will be frequent need for a gender-neutral singular pronoun. I regularly use "he" for that purpose, this being the least of the available evils. I mean no offense.

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