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Juan Pablo Pinasco

Lyapunov-type Inequalities

With Applications to Eigenvalue Problems

Springer

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To Ceci, Fede, and Selva

Preface

I used to think that the Sturm–Liouville theory of second-order ordinary differential equations was one of the most beautiful areas of mathematics. Its simplicity, together with the power of the comparison and oscillation theorems, shed a different light on second-order ordinary differential equations. However, while reading a transcription of a talk of G.C. Rota, I realized something: there are many interesting problems, both of theoretical and applied origin, that cannot be analyzed with the Sturmian tools.

Take the unit ball in R^N: just the simple reduction to polar coordinates introduces the coefficient r^{N-1}, which vanishes at the origin and is bounded above by 1, for all N. Moreover, Bessel, Hermite, Legendre, ..., almost all the special families of functions that appear as eigenfunctions of second-order ordinary differential operators, are indeed eigenfunctions of singular or degenerate operators, and the Sturmian arguments fail. What can we do now?

If we write the Sturmian bounds in modern notation, we are using the L^∞ norm of the weight, and what happens if we change it to another norm, say L^1? Indeed, the answer is known, and it is related to the stability of solutions of second-order differential equations, a problem studied by Lyapunov almost 150 years ago. He introduced an integral condition that the weight must satisfy in order to guarantee stability. However, he never proved Lyapunov's inequality. Later, Borg, Hartman, Krein, and other mathematicians working on stability gave his name to this kind of Sturmian bound with an L^1 norm.

However, unbounded domains still present a difficulty, since Lyapunov's inequality includes the length of the interval on which the problem was studied. We might decide to ignore this problem, dismissing it as a hifalutin theoretical question. But not so fast! It was, in fact, a legitimate question, inspired by quantum mechanics and related to the number of bound states of the Schrödinger equation. Ordinary differential equations on unbounded intervals were studied in the 1950s and 1960s by Jost, Pais, Bargmann, Calogero, Cohn, and Nehari (the only one who was not thinking of quantum-mechanical problems), among several others. They obtained beautiful inequalities, involving different norms of the coefficients.

And in the last twenty years, many mathematicians have extended those results to a variety of settings, including p-Laplacian operators, ordinary differential equations in Orlicz spaces, N-dimensional problems, and systems.

I designed this book as a guided tour through those results, together with their applications to eigenvalue problems, presenting full proofs and extensions of those inequalities, and showing the less-traveled paths, suggesting directions for future work. I tried to include in the references all the relevant papers on this subject, and I apologize here for the inevitable omissions.

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Buenos Aires, Argentina Juan Pablo Pinasco

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Symbols and Notation

Throughout this work,

- The letters a, b, c, d denote real numbers.
- p, q denote real numbers greater than one.
- p' denotes the Hölder conjugate of p, $p' = p/(p-1)$.
- The letters i, j, k, m, n denote a positive integer.
- λ denotes a real parameter, usually an eigenvalue of some differential operator.
- By c, C we denote positive constants, and we write $C(x, y, \ldots)$ whenever we need to stress the dependence of C on $x, y, \ldots$.
- R^N is N-dimensional Euclidean space, with $N \geq 1$.
- $\Omega \subset R^N$ is a bounded open set.
- $\partial\Omega$ is the boundary of Ω.
- u, v, w denote real-valued functions.
- Given $u : (a,b) \to R$, $u'(x)$ and $\frac{du}{dx}$ denote the derivative of u. Derivatives of higher orders are denoted by u'', $u^{(m)}$, $\frac{d^m u}{dx^m}$.
- Given $u : \Omega \to R$ with $\Omega \subset R^N$, ∇u denotes the gradient of u.
- $C_0^\infty(\Omega)$ is the space of C^∞ functions with compact support in Ω.
- If $1 \leq p \leq \infty$, $L^p(\Omega)$ denotes the usual Lebesgue space of measurable functions $u : \Omega \to R$, with the norm

$$\|u\|_p = \left(\int_\Omega |u|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

$$\|u\|_\infty = \operatorname{esssup}\{|u(x) : x \in \Omega\}.$$

- $L^1_{\mathrm{loc}}(\Omega)$ denotes the set of measurable functions integrable on compact subsets of Ω.
- $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$, $1 \leq p < \infty$, $1 \leq m$ denote the usual Sobolev spaces.