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An Introduction to Heavy-Tailed and Subexponential Distributions

Second Edition

Springer

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Preface to the First Edition

This text studies heavy-tailed distributions in probability theory, and especially convolutions of such distributions. The main goal is to provide a complete and comprehensive introduction to the theory of long-tailed and subexponential distributions which includes many novel elements and, in particular, is based on the regular use of the principle of a single big jump. Much of the material appears for the first time in text form, including:

- The establishment of new relations between known classes of subexponential distributions and the introduction of important new classes
- The development of some important new concepts, including those of h-insensitivity and local subexponentiality
- The presentation of new and direct probabilistic proofs of known asymptotic results

A number of recent textbooks and monographs contain some elements of the present theory, notably those by S. Asmussen [1, 2], P. Embrechts, C. Klüppelberg, and T. Mikosch [24], T. Rolski, H. Schmidli, V. Schmidt, and J. Teugels [47], and A. Borovkov and K. Borovkov [11]. Further, the monograph by N. Bingham, C. Goldie, and J. Teugels [9] comprehensively develops the theory of regularly varying functions and distributions; the latter form an important subclass of the subexponential distributions. We have been influenced by these books and by further contacts with their authors.

Chapters 2 and 3 of the present monograph deal comprehensively with the properties of heavy-tailed, long-tailed and subexponential distributions, and give applications to random sums. Chapter 4 develops concepts of local subexponentiality and gives further applications. Finally, Chap. 5 studies the distribution of the maximum of a random walk with negative drift and heavy-tailed increments; notably it contains new and short probabilistic proofs for the tail asymptotics of this distribution for both finite and infinite time horizons. The study of heavy-tailed distributions in more general probability models—for example, Markov-modulated models, those with dependencies, and continuous-time models—is postponed until such future date as the authors may again find some spare time. Nevertheless, the same basic principles apply there as are developed in the present text.

The authors gratefully acknowledge a fruitful collaboration on heavy-tails issues with their co-authors: Søren Asmussen, François Baccelli, Aleksandr Borovkov, Onno Boxma, Denis Denisov, Takis Konstantopoulos, Marc Lelarge, Andrew Richards, and Volker Schmidt.

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We are also very grateful both to Thomas Mikosch and to the staff of Springer for their suggestions and assistance in publishing this text.

This book was mostly written while the authors worked, together and individually, in Edinburgh and Novosibirsk; we thank our home institutions, Heriot-Watt University and the Sobolev Institute of Mathematics. A first version of this manuscript was finished while the authors stayed at the Mathematisches Forschungsinstitut Oberwolfach, under the Research in Pairs programme from March 23 to April 5, 2008; we thank the Institute for its great hospitality and support. The final version was prepared in Cambridge during our stay at the Isaac Newton Institute for Mathematical Sciences under the framework of the programme Stochastic Processes in Communication Sciences, January–June, 2010.

A list of errata and notes on further developments to this manuscript will be maintained at https://sites.google.com/site/ithtsd/ https://sites.google.com/site/ithtsd/.

Edinburgh
Novosibirsk
Oberwolfach and Cambridge

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Preface

This is an extended and corrected version of the First Edition. The major changes are:

– Chapters 2 through 5 are now appended by lists of problems and exercises. We also provide answers and a number of solutions.
– Chapter 5 includes three new sections on applications, to queueing theory, to risk, and to branching processes, and a new section describing time to exceed a high level by a random walk and its location around that time.
– Sections 5.1, 5.2 and 5.9 are extended.

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Dmitry Korshunov

August 2012

Contents

Notation and Conventions

Intervals	(x,y) is an open, $[x,y]$ a closed interval; half-open intervals are denoted by $(x,y]$ and $[x,y)$.
Integrals	$\int_x^y$ is the integral over the interval $(x,y]$.
$\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^s$	Stand for the real line, the positive real half-line $[0,\infty)$, and s-dimensional Cartesian space.
$\mathbb{Z}$, $\mathbb{Z}^+$	Stand for the set of integers and for the set $\{0,1,2,\ldots\}$.
$\mathbb{I}(A)$	Stands for the indicator function of A, i.e. $\mathbb{I}(A) = 1$ if A holds and $\mathbb{I}(A) = 0$ otherwise.
O, o, and $\sim$	Let u and v depend on a parameter x which tends, say, to infinity. Assuming that v is positive we write

$$\begin{aligned} u = O(v) &\quad \text{if} \quad \limsup_{x\to\infty} |u|/v < \infty \\ u = o(v) &\quad \text{if} \quad u/v \to 0 \text{ as } x \to \infty \\ u \sim v &\quad \text{if} \quad u/v \to 1 \text{ as } x \to \infty. \end{aligned}$$

$\mathbb{P}\{B\}$	Stands for the probability (on some appropriate space) of the event B.
$\mathbb{P}\{B\|A\}$	Stands for the conditional probability of the event B given A, i.e., for the ratio $\mathbb{P}\{BA\}/\mathbb{P}\{A\}$.
$\mathbb{E}\xi$	Stands for the mean of the random variable ξ.
$\mathbb{E}\{\xi;B\}$	Stands for the mean of ξ over the event B, i.e., for $\mathbb{E}\xi\,\mathbb{I}(B)$.
$F*G$	Stands for the convolution of the distributions F and G.
F^{*n}	Stands for the n-fold convolution of the distribution F with itself.
ξ^+, F^+	For any random variable ξ on $\mathbb{R}$ with distribution F, the random variable $\xi^+ = \max(\xi,0)$ and F^+ denotes its distribution.
:= (=:)	The quantity on the left (right) is defined to be equal to the quantity on the right (left).
□	Indicates the end of a proof.