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Functional Analysis

Fundamentals and Applications

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To the memory of my father, Robert Willem,
and to my mother, Gilberte
Willem-Groeninckx

Preface

L'induction peut être utilement employée en Analyse comme un moyen de découvertes. Mais les formules générales ainsi obtenues doivent être ensuite vérifiées à l'aide de démonstrations rigoureuses et propres à faire connaître les conditions sous lesquelles subsistent ces mêmes formules.

Augustin Louis Cauchy

Mathematical analysis leads to exact results by approximate computations. It is based on the notions of approximation and limit process. For instance, the derivative is the limit of differential quotients, and the integral is the limit of Riemann sums.

How to compute double limits? In some cases,

$$\int_\Omega \lim_{n\to\infty} u_n \, dx = \lim_{n\to\infty} \int_\Omega u_n \, dx,$$

$$\frac{\partial}{\partial x_k} \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\partial}{\partial x_k} u_n.$$

In the preceding formulas, three functional limits and one numerical limit appear. The first equality leads to the Lebesgue integral (1901), and the second to the distribution theory of Sobolev (1935) and Schwartz (1945).

In 1906, Fréchet invented an abstract framework for the limiting process: *metric spaces*. A metric space is a set X with a *distance*

$$d : X \times X \to \mathbb{R} : (u, v) \mapsto d(u, v)$$

satisfying some axioms. If the real vector space X is provided with a *norm*

$$X \to \mathbb{R} : u \mapsto ||u||,$$

then the formula

$$d(u, v) = ||u - v||$$

defines a distance on X. Finally, if the real vector space X is provided with a *scalar product*

$$X \times X \to \mathbb{R} : (u, v) \mapsto (u|v),$$

then the formula

$$||u|| = \sqrt{(u|u)}$$

defines a norm on X.

In 1915, Fréchet defined *additive functions of sets*, or *measures*. He extended the Lebesgue integral to abstract sets. In 1918, Daniell proposed a functional definition of the abstract integral. The *elementary integral*

$$\mathcal{L} \to \mathbb{R} : u \mapsto \int_{\Omega} u \, d\mu,$$

defined on a vector space $\mathcal{L}$ of *elementary functions* on Ω, satisfies certain axioms.

When u is a nonnegative μ-integrable function, its integral is given by the Cavalieri principle:

$$\int_{\Omega} u \, d\mu = \int_0^{\infty} \mu(\{x \in \Omega : u(x) > t\})dt.$$

To measure a set is to integrate its characteristic function:

$$\mu(A) = \int_{\Omega} \chi_A \, d\mu.$$

In particular, the volume of a Lebesgue-measurable subset A of $\mathbb{R}^N$ is defined by

$$m(A) = \int_{\mathbb{R}^N} \chi_A \, dx.$$

A *function space* is a space whose points are functions. Let $1 \leq p < \infty$. The real Lebesgue space $L^p(\Omega, \mu)$ with the norm

$$||u||_p = \left(\int_{\Omega} |u|^p d\mu \right)^{1/p}$$

is a *complete normed space*, or *Banach space*. The space $L^2(\Omega, \mu)$, with the scalar product

$$(u|v) = \int_{\Omega} uv \, d\mu,$$

is a *complete pre-Hilbert space*, or *Hilbert space*.

Duality plays a basic role in functional analysis. The *dual* of a normed space is the set of continuous linear functionals on this space. Let $1 < p < \infty$ and define p', the *conjugate exponent* of p, by $1/p + 1/p' = 1$. The dual of $L^p(\Omega, \mu)$ is identified with $L^{p'}(\Omega, \mu)$.

Weak derivatives are also defined by duality. Let f be a continuously differentiable function on an open subset Ω of $\mathbb{R}^N$. Multiplying $\frac{\partial f}{\partial x_k} = g$ by the *test function* $u \in \mathcal{D}(\Omega)$ and integrating by parts, we obtain

$$\int_\Omega f \frac{\partial u}{\partial x_k} dx = - \int_\Omega g\, u\, dx.$$

The preceding relation retains its meaning if f and g are locally integrable functions on Ω. If this relation is valid for every test function $u \in \mathcal{D}(\Omega)$, then by definition, g is the weak derivative of f with respect to x_k. Like the Lebesgue integral, the weak derivatives satisfy some simple double-limit rules and are used to define some complete normed spaces, the Sobolev spaces $W^{k,p}(\Omega)$.

A *distribution* is a continuous linear functional on the space of test functions $\mathcal{D}(\Omega)$. Every locally integrable function f on Ω is characterized by the distribution

$$\mathcal{D}(\Omega) \to \mathbb{R} : u \mapsto \int_\Omega f u\, dx.$$

The derivatives of the distribution f are defined by

$$\langle \frac{\partial f}{\partial x_k}, u \rangle = -\langle f, \frac{\partial u}{\partial x_k} \rangle.$$

Whereas weak derivatives may not exist, distributional derivatives always exist! In this framework, Poisson's theorem in electrostatics becomes

$$-\Delta \left(\frac{1}{|x|} \right) = 4\pi\delta,$$

where δ is the Dirac measure on $\mathbb{R}^3$.

The *perimeter* of a Lebesgue-measurable subset A of $\mathbb{R}^N$, defined by duality, is the *variation* of its characteristic function:

$$p(A) = \sup \left\{ \int_A \operatorname{div} v dx : v \in \mathcal{D}(\mathbb{R}^N; \mathbb{R}^N), \|v\|_\infty \leq 1 \right\}.$$

The space of functions of *bounded variation* $BV(\mathbb{R}^N)$ contains the Sobolev space $W^{1,1}(\mathbb{R}^N)$.

Chapter 8 contains many applications to elliptic problems and to analytic or geometric inequalities. In particular, the isoperimetric inequality and the Faber–Krahn inequality are proved by purely functional-analytic methods.

The *isoperimetric inequality* in $\mathbb{R}^N$ asserts that the ball has the largest volume among all domains with fixed perimeter. In $\mathbb{R}^2$, the isoperimetric inequality is equivalent to

$$4\pi\, m(A) \le p(A)^2.$$

The *Faber–Krahn inequality* asserts that among all domains with fixed volume, the ball has the lowest fundamental eigenvalue for the Dirichlet problem. This fundamental eigenvalue is defined by

$$\begin{aligned} -\Delta e &= \lambda_1 e \text{ in } \Omega, \\ e &> 0 \quad\;\; \text{in } \Omega, \\ e &= 0 \quad\;\; \text{on } \partial\Omega. \end{aligned}$$

Our approach is elementary and constructive. Integration theory is based on only one property: *monotone convergence*. It appears successively as an axiom, a definition, and a theorem. The inequalities of Hölder, Minkowski, and Hanner follow from the same elementary inequality, the *convexity inequality*. Weak convergence, convergence of test functions, and convergence of distributions are defined sequentially. The Hahn–Banach theorem is proved constructively in separable normed spaces and in uniformly convex smooth Banach spaces.

For the convenience of the reader, we recall the Appendix some topics in calculus. The Epilogue contains historical remarks on the close relations between functional analysis and the integral and differential calculus.

The readers must have a good knowledge of linear algebra, classical differential calculus, and the Riemann integral.

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