Undergraduate Texts in Mathematics

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Undergraduate Texts in Mathematics are generally aimed at third- and fourth-year undergraduate mathematics students at North American universities. These texts strive to provide students and teachers with new perspectives and novel approaches. The books include motivation that guides the reader to an appreciation of interrelations among different aspects of the subject. They feature examples that illustrate key concepts as well as exercises that strengthen understanding.

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Algebraic Combinatorics

Walks, Trees, Tableaux, and More

Springer

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to
Kenneth and Sharon

Preface

This book is intended primarily as a one-semester undergraduate text for a course in algebraic combinatorics. The main prerequisites are a basic knowledge of linear algebra (eigenvalues, eigenvectors, etc.) over a field, existence of finite fields, and some rudimentary understanding of group theory. The one exception is Sect. 12.6, which involves finite extensions of the rationals including a little Galois theory. Prior knowledge of combinatorics is not essential but will be helpful.

Why do I write an undergraduate textbook on algebraic combinatorics? One obvious reason is simply to gather some material that I find very interesting and hope that students will agree. A second reason concerns students who have taken an introductory algebra course and want to know what can be done with their new-found knowledge. Undergraduate courses that require a basic knowledge of algebra are typically either advanced algebra courses or abstract courses on subjects like algebraic topology and algebraic geometry. Algebraic combinatorics offers a byway off the traditional algebraic highway, one that is more intuitive and more easily accessible.

Algebraic combinatorics is a huge subject, so some selection process was necessary to obtain the present text. The main results, such as the weak Erdős–Moser theorem and the enumeration of de Bruijn sequences, have the feature that their statement does not involve any algebra. Such results are good advertisements for the unifying power of algebra and for the unity of mathematics as a whole. All but the last chapter are vaguely connected to walks on graphs and linear transformations related to them. The final chapter is a hodgepodge of some unrelated elegant applications of algebra to combinatorics. The sections of this chapter are independent of each other and the rest of the text. There are also three chapter appendices on purely enumerative aspects of combinatorics related to the chapter material: the RSK algorithm, plane partitions, and the enumeration of labelled trees. Almost all the material covered here can serve as a gateway to much additional algebraic combinatorics. We hope in fact that this book will serve exactly this purpose, that is, to inspire its readers to delve more deeply into the fascinating interplay between algebra and combinatorics.

Many persons have contributed to the writing of this book, but special thanks should go to Christine Bessenrodt and Sergey Fomin for their careful reading of portions of earlier manuscripts.

Cambridge, MA Richard P. Stanley

Contents

Basic Notation

$\mathbb{P}$	Positive integers
$\mathbb{N}$	Nonnegative integers
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers
$[n]$	The set $\{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$ (so $[0] = \emptyset$)
$\mathbb{Z}_n$	The group of integers modulo n
$R[x]$	The ring of polynomials in the variable x with coefficients in the ring R
Y^X	For sets X and Y, the set of all functions $f: X \to Y$
$:=$	Equal by definition
$\mathbb{F}_q$	The finite field with q elements
(j)	$1 + q + q^2 + \cdots + q^{j-1}$
$\#S$ or $\lvert S\rvert$	Cardinality (number of elements) of the finite set S
$S \dot\cup T$	The disjoint union of S and T, i.e., $S \cup T$, where $S \cap T = \emptyset$
2^S	The set of all subsets of the set S
$\binom{S}{k}$	The set of k-element subsets of S
$\left(\kern-.3em\binom{S}{k}\kern-.3em\right)$	The set of k-element multisets on S
KS	The vector space with basis S over the field K
B_n	The poset of all subsets of $[n]$, ordered by inclusion
$\rho(x)$	The rank of the element x in a graded poset
$[x^n]F(x)$	Coefficient of x^n in the polynomial or power series $F(x)$
$x \lessdot y$, $y \gtrdot x$	y covers x in a poset P
δ_{ij}	The Kronecker delta, which equals 1 if $i = j$ and 0 otherwise
$\lvert L\rvert$	The sum of the parts (entries) of L, if L is any array of nonnegative integers
$\ell(\lambda)$	Length (number of parts) of the partition λ

$p(n)$	Number of partitions of the integer $n \geq 0$
$\ker \varphi$	The kernel of a linear transformation or group homomorphism
$\mathfrak{S}_n$	Symmetric group of all permutations of $1, 2, \ldots, n$
ι	The identity permutation of a set X, i.e., $\iota(x) = x$ for all $x \in X$