Applied Mathematical Sciences

Volume 184

For further volumes:
http://www.springer.com/series/34

Shangjiang Guo • Jianhong Wu

Bifurcation Theory of Functional Differential Equations

Springer

Shangjiang Guo
College of Mathematics and Econometrics
Hunan University
Changsha, Hunan
China, People's Republic

Jianhong Wu
Department of Mathematics and Statistics
York University
Toronto, Ontario, Canada

ISSN 0066-5452
ISBN 978-1-4614-6991-9 ISBN 978-1-4614-6992-6 (eBook)
DOI 10.1007/978-1-4614-6992-6
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2013935530

Mathematics Subject Classification (2010): 34K18, 34C14, 34C15, 34C20, 34C23, 34C25, 35B32, 37C80, 37C85, 37G05, 37G15

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

A functional differential equation (FDE) describes the evolution of a dynamical system for which the rate of change of the state variable depends on not only the current but also the historical and even future states of the system. FDEs arise naturally in economics, life sciences, and engineering, and the study of FDEs has been a major source of inspiration for advancement of nonlinear analysis and infinite-dimensional dynamical systems. Therefore, FDEs provide an excellent theoretical platform for developing an interdisciplinary approach to understanding complex nonlinear phenomena via appropriate mathematical techniques.

Unfortunately, the study of FDEs is difficult for newcomers, since a background in nonlinear analysis, ordinary differential equations, and dynamical systems is a prerequisite. On the other hand, the novelty and challenge of fundamental research in the field of FDEs has often been underappreciated. This is especially so in our effort to describe the qualitative behaviors of solutions near equilibria or periodic orbits: these qualitative behaviors can be derived from those of finite-dimensional (ordinary differential) systems obtained through a center and center-unstable manifold reduction process, and hence the (local) bifurcation theory that deals with significant changes in these qualitative behaviors is in principle a consequence of the corresponding theory for finite-dimensional (ordinary differential) systems. The highly nontrivial and often lengthy calculation of center manifold reduction, however, not only leads to enormous duplication of calculation efforts, but also prevents us from discovering simple and key mechanisms behind observed bifurcation phenomena due to the infinite-dimensionality of FDEs. This, in turn, makes it difficult to express bifurcation results explicitly in terms of model parameters and to compare and validate different results. Another challenge is the study of the birth and global continuation of bifurcation of periodic solutions and the coexistence of multiple periodic solutions when the parameters are far from the bifurcation/critical values. There has been substantial progress dedicated to this global bifurcation problem, and remarkably, the presence of a delayed or advanced argument in the nonlinearity can sometimes facilitate the application of topological methods such as equivalent degrees to examine the global continua of branches of periodic solutions, and this has inspired interesting developments in the spectral analysis of circulant matrices.

On the other hand, the study of dynamical systems with symmetries has become well established as a major branch of nonlinear systems theory. The current interest in the field dates mainly to the appearance of the equivariant branching lemma of Vanderbauwhede and Cicogna and the equivariant Hopf bifurcation theorem of Golubitsky and Stewart, both of which are reviewed in the book by Golubitsky, Stewart and Schaeffer. Since then, important new theories have been developed for more complex dynamical phenomena, including the existence, stability, and bifurcations of systems of heteroclinic connections, and the symmetry groups and bifurcations of chaotic attractors.

To a large extent, the phenomenal growth in the subject has been due to its effectiveness in explaining the bifurcations and dynamical phenomena that are seen in a wide range of physical systems including coupled oscillators, reaction–diffusion systems, convecting fluids, and mechanical systems. A local symmetric bifurcation theory for FDEs can be derived from that of but since some special properties of spatiotemporal symmetry of FDEs may be reflected generically in the reduced finite-dimensional systems, one can and should make general observations about the particular bifurcation patterns of symmetric FDEs.

The purpose of this book is to summarize some practical and general approaches and frameworks for the investigation of bifurcation phenomena of FDEs depending on parameters. The book aims to be self-contained, so the reader should find in this book all relevant materials on bifurcation, dynamical systems with symmetry, functional differential equations, normal forms, and center manifold reduction. This material was used in graduate courses on functional differential equations at Hunan University (China) and York University (Canada). We want to thank all students in these courses for their careful reading and some helpful comments. We would like especially to thank Dr. Jing Fang and Dr. Xiang-Sheng Wang for their careful reading of an early version of the manuscript and for their critical comments.

This work was supported in part by the National Natural Science Foundation (China), the Program for New Century Excellent Talents in University of Ministry of Education (China), the Research Fund for the Doctoral Program of Higher Education of China, the Hunan Provincial Natural Science Foundation, the NCE Centre Mathematics for Information Technology and Complex Systems, Mprime, the Canada Research Chairs Program, and the Natural Sciences and Engineering Research Council of Canada.

Changsha, Hunan, China — Shangjiang Guo
Toronto, ON, Canada — Jianhong Wu

Contents

1 Introduction to Dynamic Bifurcation Theory 1
1.1 Introduction 1
1.2 Topological Equivalence 2
1.3 Structural Stability 4
1.4 Codimension-One Bifurcations of Equilibria 6
1.4.1 Fold Bifurcation 8
1.4.2 Poincaré–Andronov–Hopf Bifurcation 10
1.5 Transcritical and Pitchfork Bifurcations of Equilibria 12
1.6 Bifurcations of Closed Orbits 15
1.7 Homoclinic Bifurcation 17
1.8 Heteroclinic Bifurcation 19
1.9 Two-Parameter Bifurcations of Equilibria 20
1.9.1 Bogdanov–Takens Bifurcation 21
1.9.2 Cusp Bifurcation 25
1.9.3 Fold–Hopf Bifurcation 27
1.9.4 Bautin Bifurcation 33
1.9.5 Hopf–Hopf Bifurcation 35
1.10 Some Other Bifurcations 40

2 Introduction to Functional Differential Equations 41
2.1 Infinite Dynamical Systems Generated by Time Lags 41
2.2 The Framework for DDEs 44
2.2.1 Definitions 44
2.2.2 An Operator Equation 45
2.2.3 Spectrum of the Generator 47
2.2.4 An Adjoint Operator 48
2.2.5 A Bilinear Form 49
2.2.6 Neural Networks with Delay: A Case Study on Characteristic Equations 50
2.3 General Framework of NFDEs 58

3 Center Manifold Reduction ... 61
3.1 Some Examples of Ordinary Differential Equations ... 61
3.2 Invariant Manifolds of RFDEs ... 65
3.3 Center Manifold Theorem ... 66
3.4 Calculation of Center Manifolds ... 68
3.4.1 The Hopf Case ... 69
3.4.2 The Fold–Hopf Case ... 72
3.4.3 The Double Hopf Case ... 74
3.5 Center Manifolds with Parameters ... 76
3.6 Preservation of Symmetry ... 82

4 Normal Form Theory ... 85
4.1 Introduction ... 85
4.2 Unperturbed Vector Fields ... 86
4.2.1 The Poincaré–Birkhoff Normal Form Theorem ... 87
4.2.2 Computation of Normal Forms ... 89
4.2.3 Internal Symmetry ... 96
4.3 Perturbed Vector Fields ... 103
4.3.1 Normal Form for Hopf Bifurcation ... 103
4.3.2 Norm Form Theorem ... 106
4.3.3 Preservation of External Symmetry ... 108
4.4 RFDEs with Symmetry ... 110
4.4.1 Basic Assumptions ... 110
4.4.2 Computation of Symmetric Normal Forms ... 112
4.4.3 Nonresonance Conditions ... 116

5 Lyapunov–Schmidt Reduction ... 119
5.1 The Lyapunov–Schmidt Method ... 119
5.2 Derivatives of the Bifurcation Equation ... 121
5.3 Equivariant Equations ... 122
5.4 The Steady-State Equivariant Branching Lemma ... 123
5.5 Generalized Hopf Bifurcation of RFDE ... 125
5.6 Equivariant Hopf Bifurcation of NFDEs ... 134
5.7 Application to a Delayed van der Pol Oscillator ... 141
5.8 Applications to a Ring Network ... 145
5.9 Coupled Systems of NFDEs and Lossless Transmission Lines ... 147
5.10 Wave Trains in the FPU Lattice ... 150

6 Degree Theory ... 153
6.1 Introduction ... 153
6.2 The Brouwer Degree ... 154
6.3 The Leray–Schauder Degree ... 156
6.4 Global Bifurcation Theorem ... 157
6.5 $\mathbb{S}^1$-Equivariant Degree ... 158
6.5.1 Differentiability Case ... 161
6.5.2 Nondifferentiability Case ... 165

6.6 Global Hopf Bifurcation Theory of DDEs 167
6.7 Application to a Delayed Nicholson Blowflies Equation 172
6.7.1 The Nicholson Blowflies Equation 172
6.7.2 The Global Hopf Bifurcation Theorem of Wei–Li 172
6.7.3 Nicholson's Blowflies Equation Revisited: Onset and Termination of Nonlinear Oscillations 178
6.8 Rotating Waves and Circulant Matrices 179
6.9 State-Dependent DDEs 188
6.9.1 Local Hopf Bifurcation 190
6.9.2 Global Bifurcation 199
6.9.3 Uniform Bounds for Periods of Periodic Solutions in a Connected Component 204
6.9.4 Uniform Boundedness of Periodic Solutions 217
6.9.5 Global Continuation of Rapidly Oscillating Periodic Solutions: An Example 224

7 Bifurcation in Symmetric FDEs 231
7.1 Introduction 231
7.2 Fold Bifurcation 232
7.2.1 Standard Fold Bifurcation 232
7.2.2 Fold Bifurcations with $\mathbb{Z}_2$-Symmetry 235
7.2.3 Fold Bifurcations with O(2)-Symmetry 236
7.3 Hopf Bifurcation 237
7.3.1 A Little History 238
7.3.2 Standard Hopf Bifurcation 239
7.3.3 Equivariant Hopf Bifurcation 242
7.3.4 Application to $\mathbb{D}_n$-Equivariant Hopf Bifurcation 244
7.3.5 Hopf Bifurcation in a Ring Network 251
7.4 Bogdanov–Takens Bifurcation 259
7.4.1 Center Manifold Reduction 259
7.4.2 Bogdanov Normal Form 261
7.4.3 Normal Form of System (7.60) with a Fixed Equilibrium 263
7.4.4 $\mathbb{D}_3$-Equivariant Bogdanov–Takens Bifurcation 264
7.5 Double Hopf Bifurcation 268

References 275

Index 287