Part 3. Trace Formulae

The work of Part 4 depends on a comparison of the fixed point formula and the trace formula. Since only automorphic $G(\mathbb{A})$-modules occur in the Selberg formula, the purpose of this approach is to show that the $G(\mathbb{A}_f)$-modules $\widetilde{\pi}_f$ which occur in the virtual module $H_c^* = \sum_i (-1)^i H_c^i$ are automorphic, in addition to establishing the relation concerning the local Frobenius and Hecke eigenvalues. The Grothendieck fixed point formula gives an expression for the trace of the action of the (geometric) Frobenius $\mathrm{Fr}_v \times 1$ on the cohomology module H_c^* by means of the set of points in $M_{r,I,v}(\overline{\mathbb{F}}_v)$ fixed by the action of the Frobenius and the traces of the resulting morphisms on the stalks of the $\overline{\mathbb{Q}}_\ell$-sheaf $\mathbb{L}(\rho)$ at the fixed points.

Part 3 prepares for the comparison. Following [D2], in Chap. 7 the set $M_{r,I,v}(\overline{\mathbb{F}}_v)$ is expressed as a disjoint union of isogeny classes of elliptic modules over $\overline{\mathbb{F}}_v$, and their types are studied. In Chap. 8 it is shown that the elliptic modules with level structure of a given type make a homogeneous space under the action of $G(\mathbb{A}_f)$, and the stabilizer is described. Moreover, the action of the Frobenius Fr_v is identified with multiplication by a certain matrix. A *type* is described in group theoretic terms of an elliptic torus in $G(F)$ (see Definition 7.3), and the cardinality of the set $M_{r,I,v}(\mathbb{F}_{v,n})$ ($[\mathbb{F}_{v,n} : \mathbb{F}_v] = n$) is expressed in terms of orbital integrals of conjugacy classes γ in $G(F)$ which are elliptic in $G(F_\infty)$ and n-admissible (see Definition 8.1) at v.

Next, in Chap. 9, it is shown that the orbital integral at v obtained in Chap. 8 can be expressed as an orbital integral of a spherical function $f_n = f_n^{(r)}$ on G_v whose normalized orbital integral $F(f_n)$ is supported on the n-admissible set. This spherical function is defined by the relation $\mathrm{tr}(\pi_v(z))(f_n) = q_v^{n(r-1)/2} \sum_{i=1}^r z_i^n$.