Part 2. Hecke Correspondences

In this part and in Parts 3 and 4 we study some relations between two natural actions—of a Galois group and of a Hecke algebra—on ℓ-adic cohomology groups with compact support and coefficients in a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathbb{L}(\rho)$, attached to the geometric generic fiber $M_{r,I} \otimes_A \overline{F}$ of the moduli scheme $M_{r,I}$ constructed in Part 1, Chap. 4. We present two approaches. That of Part 2, Sect. 6.2, is based on congruence relations and Hecke correspondences. That of Part 4 is an application of the trace formula. Chapters 7–9 of Part 3 develop the tools needed for the comparison in Chaps. 10–11 of Part 4 of the Grothendieck–Lefschetz fixed point formula with the Selberg trace formula. We begin with a summary (in Sect. 6.1) of those properties of the ℓ-adic cohomology groups with compact support which we need in order to state—and use in the proof of—our main theorems in Chaps. 6, 10, and 11. The main application of the theory of congruence relations is given in Theorem 6.10, which asserts that if $\pi_f \times \sigma$ is an irreducible $G(\mathbb{A}_f) \times \mathrm{Gal}(\overline{F}/F)$-subquotient of $H^i_c(M_{r,I} \times_A \overline{F}, \mathbb{L}(\rho))$, then for almost all v each eigenvalue of the geometric Frobenius endomorphism $\sigma(\mathrm{Fr}_v \times 1)$ at v is the product by $q_v^{(r-1)/2}$ of a Hecke eigenvalue of the component π_v of π at v. Consequently $\sigma(\mathrm{Fr}_v \times 1)$ has at most r distinct eigenvalues.