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Introduction to Perturbation Methods

Second Edition

Springer

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To Colette, Matthew, and Marianna
A small family with big hearts

Preface

First, let me say hello and welcome to the subject of perturbation methods. For those who may be unfamiliar with the topic, the title can be confusing. The first time I became aware of this was during a family reunion when someone asked what I did as a mathematician. This is not an easy question to answer, but I started by describing how a certain segment of the applied mathematics community was interested in problems that arise from physical problems. Examples such as water waves, sound propagation, and the aerodynamics of airplanes were discussed. The difficulty of solving such problems was also described in exaggerated detail. Next came the part about how one generally ends up using a computer to actually find the solution. At this point I editorialized on the limitations of computer solutions and why it is important to derive, if at all possible, accurate approximations of the solution. This led naturally to the mentioning of asymptotics and perturbation methods. These terms ended the conservation because I was unprepared for people's reactions. They were not sure exactly what asymptotics meant, but they were quite perplexed about perturbation methods. I tried, unsuccessfully, to explain what it means, but it was not until sometime later that I realized the difficulty. For them, as in *Webster's Collegiate Dictionary*, the first two meanings for the word perturb are "to disturb greatly in mind (disquiet); to throw into confusion (disorder)." Although a cynic might suggest this is indeed appropriate for the subject, the intent is exactly the opposite. For a related comment, see Exercise 3.18(d).

In a nutshell, this book serves as an introduction to systematically constructing an approximation of the solution to a problem that is otherwise intractable. The methods all rely on there being a parameter in the problem that is relatively small. Such a situation is relatively common in applications, and this is one of the reasons that perturbation methods are a cornerstone of applied mathematics. One of the other cornerstones is scientific computing, and it is interesting that the two subjects have grown up together. However, this is not unexpected given their respective capabilities. Using a computer one can solve problems that are nonlinear, inhomogeneous, and multidimensional. Moreover, it is possible to achieve very high accuracy. The drawbacks are that computer solutions do not provide much insight into the physics of problems (particularly when one does not have access to the appropriate software or computer), and there is always the question as to whether or not the computed solution is correct. On the other hand, perturbation methods are also capable of dealing with nonlinear, inhomogeneous, and multidimensional problems (although not to the same extent as computer-generated solutions). The principal objective when using perturbation methods, at least as far as the author is concerned, is to provide a reasonably accurate expression for the solution. In doing this one can derive an understanding of the physics of a problem. Also, one can use the result, in conjunction with the original problem, to obtain more efficient numerical procedures for computing the solution.

The methods covered in the text vary widely in their applicability. The first chapter introduces the fundamental ideas underlying asymptotic approximations. This includes their use in constructing approximate solutions of transcendental equations as well as differential equations. In the second chapter, matched asymptotic expansions are used to analyze problems with layers. Chapter 3 describes a method for dealing with problems with more than one time scale. In Chap. 4, the WKB method for analyzing linear singular perturbation problems is developed, while in Chap. 5 a method for dealing with materials containing disparate spatial scales (e.g., microscopic vs. macroscopic) is discussed. The last chapter examines the topics of multiple solutions and stability.

The mathematical prerequisites for this text include a basic background in differential equations and advanced calculus. In terms of difficulty, the chapters are written so the first sections are either elementary or intermediate, while the later sections are somewhat more advanced. Also, the ideas developed in each chapter are applied to a spectrum of problems, including ordinary differential equations, partial differential equations, and difference equations. Scattered through the exercises are applications to integral equations, integrodifferential equations, differential-difference equations, and delay equations. What will not be found is an in-depth discussion of the theory underlying the methods. This aspect of the subject is important, and references to the more theoretical work in the area are given in each chapter.

The exercises in each section vary in their complexity. In addition to the more standard textbook problems, an attempt has been made to include problems from the research literature. The latter are intended to provide a window into the wide range of areas that use perturbation methods. Solutions to some of the exercises are available and can be obtained, at no charge, from the author's home page. Also included, in the same file, is an errata sheet. Readers who would like to make a contribution to this file or who have suggestions about the text can reach the author at holmes@rpi.edu.

I would like to express my gratitude to the many students who took my course in perturbation methods at Rensselaer. They helped me immeasurably in understanding the subject and provided much needed encouragement to write this book. It is a pleasure to acknowledge the suggestions of Jon Bell, Ash Kapila, and Bob O'Malley, who read early versions of the manuscript. I would also like to thank Julian Cole, who first introduced me to perturbation methods and is still, to this day, showing me what the subject is about.

Troy, NY Mark H. Holmes

Preface to the Second Edition

It's interesting reading something you wrote 15 years earlier, not just because of what you *did* write but also because of what you did *not* write. You also realize how the subject has evolved, that certain topics should be rewritten and others included. It is for these reasons that this second edition was undertaken. As will be explained in the next paragraph, every section has been edited, many only in minor ways, while others have been completely revised; new material has also been added. This includes approximations for weakly coupled oscillators, analysis of problems that involve transcendentally small terms, an expanded discussion of Kummer functions, and metastability. Also, one of the core objectives of this book is to develop the ideas underlying perturbation methods and then demonstrate how they can be used in a wide variety of problems. To provide background on some of these areas, two appendices have been added, one on solving difference equations, the other on delay equations. Finally, a few things have also been removed, the most prominent of which is the appendix containing the numerical solution of boundary-value problems. The code given in the earlier edition is available at the author's home page and is also discussed at length in Holmes (2007). Finally, the references have been updated, new exercises added, and a few of the original exercises modified.

There is an interesting aside that is worth telling. The first edition was written using a commercial software program. At the time, Springer published their books using LaTeX, so they needed to use a translator program to convert the manuscript. With this, they had to redraw, by hand, all of the figures. They also had to write macros to produce some of the lettering used for balancing equations. The result was very nice. Unfortunately, none of this worked for this edition. The original files cannot be used with the current version of the commercial software program, and the converted LaTeX files exist only in fragments. Consequently, this edition was written practically from scratch, with every figure redrawn. With this effort come a few benefits. One is that the manuscript is now in LaTeX and should be useable for the foreseeable future. Second, many of the figures were redrawn using MATLAB, and the codes used for the figures are available at the author's home page. You might find them useful, either for teaching a class or else for the insights they provide into how to proceed when working out the homework problems. You might also enjoy the videos available, which show some of the solutions of the time-dependent problems solved in the book. Also, there is an errata page as well as answers to some of the exercises.

I would like to thank those who developed and have maintained TeXShop, a free and very good TeX previewer.

Troy, New York Mark H. Holmes

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