

# Texts in Applied Mathematics

## Volume 51

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# Computational Electromagnetics

Second Edition

 Springer

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*To the memory of Anders Bondeson*



# Foreword

The material in this book was developed for an undergraduate course in computational electromagnetics, initially given by Professor Anders Bondeson at Chalmers University of Technology, Göteborg, Sweden. It was used in various forms for almost a decade and it fostered a number of engineers who today work in academia and industry with computational electromagnetics as a main tool. During this time, we never managed to find any textbooks suitable for an introductory course on the subject matter, so we eventually decided on compiling our material into a book. The first edition of the book was published in 2005 and now, seven years later, the second edition is completed with new material that hopefully makes the book even more useful. In particular, we wish to thank students, readers and reviewers around the world for contributing with important feedback on the book.

On the 20th of March 2004, before the first edition of the book was completed, Professor Anders Bondeson passed away, suddenly and unexpectedly. This caused shock and deep sorrow to all of us who worked with him, and naturally interrupted the creation of the book. Nevertheless, we felt we should complete the first edition of the book, and we would like to thank the publisher Springer for all the understanding, support, and encouragement during the difficult time after the passing of our coauthor, colleague, friend and former advisor. In particular, we would like to thank Donna Chernyk, Achi Dosanjh, Jamie Ehrlich, Yana Mermel, Brian Halm, and Frank McGuckin at Springer. We are also grateful to the anonymous reviewers who contributed with valuable comments on the manuscript.

We dedicate this book to the memory of Anders Bondeson.

Göteborg

Thomas Rylander  
Pär Ingelström





# Preface to the Second Edition

The first edition of this book was published 7 years ago, and since then it has been used for courses and self-study around the world. We have received many useful comments on the presentation of the material and the contents of the book from students, lecturers, researchers, and other readers. In addition, we have used the book ourselves for various courses that we teach on computational electromagnetics (CEM), where we received direct feedback from our students on the book as a source of information and its pedagogical development of the topic. Given the collection of all comments that we received in combination with our own experience, we have complemented the second edition of the book with material that we believe will help the reader to learn the subject matter. In particular, we have strengthened the discussion concerning numerical techniques for the first-order system of Maxwell's equations and, as a consequence, their relation to the corresponding second-order differential equations. For finite-difference schemes, this is manifested in terms of the staggered grids that are used to represent the electric and magnetic fields, where particular emphasis is placed on analysis with complex exponentials. In the context of finite element methods (FEMs), the first-order system is treated by means of expanding the electric field in terms of curl-conforming elements and the magnetic field in terms of divergence-conforming elements. This representation associates the electric field with edges (referred to by some authors as a primal grid), and, similarly, the magnetic field is associated with faces (which may be thought of as a dual grid). In particular, for brick-shaped elements it is rather apparent that the primal grid and the dual grid together make up the staggered grids that are used for finite-difference schemes. Furthermore, we have incorporated a new Appendix B that features the lowest-order curl-conforming and divergence-conforming basis functions on the most common finite element shapes: the triangle and the quadrilateral, the tetrahedron, the prism, and the hexahedron. The automatic generation of symbolic expressions and vector field visualizations for these basis functions is provided in terms of a MATLAB implementation that can be downloaded from a URL provided at the end of this preface. The presentation of the material is further improved by cross references between the description of the finite difference schemes and the FEMs. In addition, the presentation of both finite differences and finite elements

references the material in the new Appendix B, which may be useful for obtaining a unified perspective on CEM. In a similar fashion, the chapter on the method of moments (MoM) exploits this new Appendix B.

Appendix B on the lowest-order curl-conforming and divergence-conforming basis functions also contains basis functions for nodal elements (i.e., gradient-conforming basis functions), a definition of reference elements, and suitable quadrature rules. The concept of the reference element is useful for the implementation of the FEM (and MoM), where the reference element is related to the physical element by means of a mapping. The mapping is expressed in terms of the nodal basis functions, and the evaluation of discrete operators is formulated in terms of numerical integration (or quadrature) on the reference element. This type of approach is at the heart of most finite element programs and is particularly useful for problems that involve inhomogeneous materials or nonlinear materials. The second edition of the book contains a new Sect. 6.6 that describes in detail the concept of the reference element and its relation to the physical element by means of a mapping, where integration on the reference element is also featured. These concepts are presented with the triangle as an example, and references to the new Appendix B are used extensively to demonstrate how the techniques generalize to other element shapes. In addition, the new Appendix B is very useful as a source of information for computer implementations, where the reader can select and combine different discretization techniques that are suitable for the specific situation at hand. We have also added a new advanced real-world problem in Sect. 6.6.4 that demonstrates the use of all these techniques, where we solve a 3D eigenvalue problem formulated in terms of the FEM on tetrahedrons applied to the first-order system of Maxwell's equations with inhomogeneous and lossy media. The theoretical formulation is described in detail, and the complete computer implementation is included in the material that can be downloaded from the Internet, at the URL provided at the end of this preface. Also, we present some results in terms of 3D visualizations for a circular cylindrical cavity of finite length and discuss their relation to the conventional analysis. In addition, we solve a more challenging problem with inhomogeneous and lossy media in a cavity. The reader can easily modify the computer implementation to study, e.g., driven problems in the frequency domain or exploit the computer implementation provided here as a platform for more advanced programs with other features such as radiation boundary conditions, transient analysis, and nonlinear media. The book also contains material on numerical methods that exploit higher-order approximations, and this is demonstrated in particular in Sect. 3.3, which provides a method for deriving higher-order finite-difference approximations.

The second edition of this book also features a new Appendix A with five computer projects: (1) convergence and extrapolation, (2) finite differences in the frequency domain, (3) finite-difference time-domain scheme, (4) FEM, and (5) MoM. This collection of projects covers the material presented in the book, and the projects are designed to progress the knowledge and skills of the reader. They are useful for learning and understanding the material in the book and could be used as assignments in a course. Such assignments could be examined in several different ways such as (1) oral examination in a computer-laboratory setting, (2)

presentation in a classroom setting where students and teacher ask questions, or (3) student-authored written reports. Experience shows that it is useful to let students work in small groups. This approach gives the students the possibility to present, challenge, and discuss different ideas on how to work on the assignments.

To our knowledge, the open literature is sparse on books that present contemporary CEM in an introductory manner that is appropriate for use at the undergraduate level. This book is intended to provide such material and prepare the reader for the more advanced literature on CEM. Apart from this text, the book by Davidson [22] and the book by Sadiku [68] are some of the very few other examples with such an aim. Additional books that present the main computational techniques in CEM have been published in recent years. Those by Garg [29] and Jin [42] contain a fair amount of classical analytical calculations for Maxwell's equations apart from the basic techniques in CEM. Warnick [88] and Sheng [72] have a stronger focus on CEM; in addition, Warnick [88] also discusses numerical techniques in somewhat more general settings.

The MATLAB implementations given in this book, together with the MATLAB implementations that support the projects in Appendix A and some other useful programs, are available for download at <http://www.springer.com>. We would appreciate it if errors found were brought to our attention. These will be posted on the aforementioned Web site.



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# List of Symbols

Symbol	Quantity	SI Unit
$E$	Electric field strength	V/m
$D$	Displacement field strength	C/m <sup>2</sup>
$H$	Magnetic field strength	A/m
$B$	Magnetic flux	T = Vs/m <sup>2</sup>
$J$	Electric current density	A/m <sup>2</sup>
$\rho$	Electric charge density	C/m <sup>3</sup>
$\epsilon$	Electric permittivity	As/Vm
$\mu$	Magnetic permeability	Vs/Am
$\sigma$	Electric conductivity	S/m = A/Vm
$t$	time	s
$\omega$	angular frequency	rad/s



# List of Acronyms

ABC	Absorbing Boundary Conditions
BEM	Boundary Element Method
CEM	Computational Electromagnetics
CFIE	Combined Field Integral Equation
CG	Conjugate Gradient
DWP	Discrete Wavelet Packet
EMC	Electromagnetic Compatibility
FD	Finite Difference
FDTD	Finite-Difference Time-Domain
FEM	Finite Element Method
FMM	Fast Multipole Method
FIT	Finite Integration Technique
FV	Finite Volume
IML	Impedance Matrix Localization (technique)
MFIE	Magnetic Field Integral Equation
MLFMA	Multilevel Fast Multipole Algorithm
MoM	Method of Moments
NTF	Near-to-far field transformation
ODE	Ordinary Differential Equation
PEC	Perfect electric conductor
PML	Perfectly Matched Layer
RCS	Radar Cross Section
RWG	Rao-Wilton-Glisson (boundary elements)
TDIE	Time-Domain Integral-Equation
TE	Transverse Electric
TLM	Transmission Line Method
TM	Transverse Magnetic
TEM	Transverse Electromagnetic

