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Topics in Fractional Differential Equations

 Springer

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We dedicate this book to our family members. In particular, Saïd Abbas dedicates to the memory of his father, to his mother, his wife Zoubida and his children Mourad, Amina, and Ilyes; Mouffak Benchohra makes his dedication to the memory of his father Yahia Benchohra and Gaston N'Guérékata to the memory of his father Jean N'Guérékata.

Preface

Fractional calculus (FC) generalizes integrals and derivatives to non-integer orders. During the last decade, FC was found to play a fundamental role in the modeling of a considerable number of phenomena, in particular, the modeling of memory-dependent phenomena and complex media such as porous media. FC emerged as an important and efficient tool for the study of dynamical systems where classical methods reveal strong limitations. This book is devoted to the existence and uniqueness of solutions for various classes of Darboux problem for hyperbolic differential equations or inclusions involving the Caputo fractional derivative, the best fractional derivative of the time. Some equations present delay which may be finite, infinite, or state-dependent. Others are subject to impulsive effect. The tools used include classical fixed point theorems as well as sharp (new) ones such as the one by Dhage on ordered Banach algebras and the fixed point theorem for contraction multivalued maps due to Covitz and Nadier, as well as some generalizations of the Gronwall's lemma. Each chapter concludes with a section devoted to notes and bibliographical remarks and all abstract results are illustrated by examples.

The content of this book is new and complements the existing literature in fractional calculus. It is useful for researchers and graduate students for research, seminars, and advanced graduate courses, in pure and applied mathematics, engineering, biology, and all other applied sciences.

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