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Rodney Coleman

Calculus on Normed Vector Spaces

Springer

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ISSN 0172-5939 ISSN 2191-6675 (electronic)
ISBN 978-1-4614-3893-9 ISBN 978-1-4614-3894-6 (eBook)
DOI 10.1007/978-1-4614-3894-6
Springer New York Heidelberg Dordrecht London

Library of Congress Control Number: 2012939881

Mathematics Subject Classification (2010): 46–XX, 46N10

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

To Francie, Guillaume, Samuel and Jeremy

Preface

The aim of this book is to present an introduction to calculus on normed vector spaces at a higher undergraduate or beginning graduate level. The prerequisites are basic calculus and linear algebra. However, a certain mathematical maturity is also desirable. All the necessary topology and functional analysis is introduced where necessary.

I have tried to show how calculus on normed vector spaces extends the basic calculus of functions of several variables. I feel that this is often not done and we have, on the one hand, very elementary texts, and on the other, high level texts, but few bridging the gap.

In the text there are many nontrivial applications of the theory. Also, I have endeavoured to give exercises which seem, at least to me, interesting. In my experience, very often the exercises in books are trivial or very academic and it is difficult to see where the interest lies (if there is any!).

In writing this text I have been influenced and helped by many other works on the subject and by others close to it. In fact, there are too many to mention; however, I would like to acknowledge my debt to the authors of these works. I would also like to express my thanks to Mohamed El Methni and Sylvain Meignen, who carefully read the text and gave me many helpful suggestions.

Writing this book has allowed me to clarify many of my ideas and it is my sincere hope that this work will prove useful in aiding others.

Grenoble, France — Rodney Coleman

Contents

Preface vii

1 Normed Vector Spaces 1
- 1.1 First Notions 1
- 1.2 Limits and Continuity 4
- 1.3 Open and Closed Sets 7
- 1.4 Compactness 11
- 1.5 Banach Spaces 16
- 1.6 Linear and Polynomial Mappings 19
- 1.7 Normed Algebras 24
- 1.8 The Exponential Mapping 28
- Appendix: The Fundamental Theorem of Algebra 31

2 Differentiation 35
- 2.1 Directional Derivatives 35
- 2.2 The Differential 37
- 2.3 Differentials of Compositions 43
- 2.4 Mappings of Class C^1 45
- 2.5 Extrema 50
- 2.6 Differentiability of the Norm 54
- Appendix: Gâteaux Differentiability 59

3 Mean Value Theorems 61
- 3.1 Generalizing the Mean Value Theorem 62
- 3.2 Partial Differentials 66
- 3.3 Integration 69
- 3.4 Differentiation under the Integral Sign 75

4 Higher Derivatives and Differentials 79
- 4.1 Schwarz's Theorem 80
- 4.2 Operations on C^k-Mappings 83
- 4.3 Multilinear Mappings 84
- 4.4 Higher Differentials 89

4.5 Higher Differentials and Higher Derivatives 93
4.6 Cartesian Product Image Spaces 96
4.7 Higher Partial Differentials 97
4.8 Generalizing C^k to Normed Vector Spaces 98
4.9 Leibniz's Rule 101

5 Taylor Theorems and Applications 107
5.1 Taylor Formulas 107
5.2 Asymptotic Developments 113
5.3 Extrema: Second-Order Conditions 115
Appendix: Homogeneous Polynomials 121

6 Hilbert Spaces 125
6.1 Basic Notions 125
6.2 Projections 130
6.3 The Distance Mapping 136
6.4 The Riesz Representation Theorem 138

7 Convex Functions 141
7.1 Preliminary Results 141
7.2 Continuity of Convex Functions 144
7.3 Differentiable Convex Functions 147
7.4 Extrema of Convex Functions 152
Appendix: Convex Polyhedra 155

8 The Inverse and Implicit Mapping Theorems 161
8.1 The Inverse Mapping Theorem 161
8.2 The Implicit Mapping Theorem 165
8.3 The Rank Theorem 170
8.4 Constrained Extrema 177
Appendix 1: Bijective Continuous Linear Mappings 185
Appendix 2: Contractions 188

9 Vector Fields 189
9.1 Existence of Integral Curves 192
9.2 Initial Conditions 199
9.3 Geometrical Properties of Integral Curves 205
9.4 Complete Vector Fields 206
Appendix: A Useful Result on Smooth Functions 210

10 The Flow of a Vector Field 213
10.1 Continuity of the Flow 213
10.2 Differentiability of the Flow 214
10.3 Higher Differentiability of the Flow 220
10.4 The Reduced Flow 223
10.5 One-Parameter Subgroups 226

11 The Calculus of Variations: An Introduction 229
11.1 The Space $C^1(I, E)$ 229
11.2 Lagrangian Mappings 230
11.3 Fixed Endpoint Problems 233
11.4 Euler–Lagrange Equations 236
11.5 Convexity 240
11.6 The Class of an Extremal 241

References 245

Index 247