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Rodney Coleman

# Calculus on Normed Vector Spaces

 Springer

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*To Francie, Guillaume, Samuel  
and Jeremy*



# Preface

The aim of this book is to present an introduction to calculus on normed vector spaces at a higher undergraduate or beginning graduate level. The prerequisites are basic calculus and linear algebra. However, a certain mathematical maturity is also desirable. All the necessary topology and functional analysis is introduced where necessary.

I have tried to show how calculus on normed vector spaces extends the basic calculus of functions of several variables. I feel that this is often not done and we have, on the one hand, very elementary texts, and on the other, high level texts, but few bridging the gap.

In the text there are many nontrivial applications of the theory. Also, I have endeavoured to give exercises which seem, at least to me, interesting. In my experience, very often the exercises in books are trivial or very academic and it is difficult to see where the interest lies (if there is any!).

In writing this text I have been influenced and helped by many other works on the subject and by others close to it. In fact, there are too many to mention; however, I would like to acknowledge my debt to the authors of these works. I would also like to express my thanks to Mohamed El Methni and Sylvain Meignen, who carefully read the text and gave me many helpful suggestions.

Writing this book has allowed me to clarify many of my ideas and it is my sincere hope that this work will prove useful in aiding others.

Grenoble, France

Rodney Coleman





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