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Hamiltonian Structures and Generating Families

Springer

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Everything has a generating family.

Preface

A *Hamiltonian structure* is a mathematical model of a physical phenomenon in which symplectic geometry plays a basic role. The Hamiltonian formulation of analytical mechanics as well as the Hamiltonian formulation of geometrical optics, place of birth of the Hamilton–Jacobi equation, are well-known examples. Other examples can be added, for instance, the control of static mechanical systems and of the equilibrium states of thermodynamic systems.

A *generating family* is a smooth real function that is able to describe special subsets of a cotangent bundle, here called *Lagrangian sets*. A Lagrangian set may be a *Lagrangian submanifold*. However, as we show, several examples of physically meaningful phenomena are in fact represented by Lagrangian sets that are not submanifolds.

The sense of this dichotomy, *nonsmooth* and *smooth*, becomes clear when we deal with *symplectic relations*, one of the most important tools used in this book. A symplectic relation is defined, at a first stage, as a Lagrangian submanifold of the product of two symplectic manifolds. If these symplectic manifolds are cotangent bundles, then a symplectic relation has (locally or globally) a generating family. Relations can be composed according to a well-defined rule, but the composition of two smooth relations (i.e., submanifolds) may not be smooth; that is, a Lagrangian subset of the product of two cotangent bundles. However, besides the composition of symplectic relations, we have a composition rule of their generating families which yields another smooth generating function. In other words, although the composition of two symplectic relations may produce a nonsmooth object, the composition of their generating families is always smooth. Then, the *symplectic creed* formulated by Alan Weinstein in his article "Symplectic geometry" (1981)

everything is a Lagrangian submanifold,

which means that one should try to express objects in symplectic geometry and mechanics in terms of Lagrangian submanifolds, is here replaced by

everything has a generating family.

In order to make this book self-contained and to clarify the notations, the first two chapters are devoted to those basic notions of calculus on manifolds that are strictly necessary to our purposes.

Chapter 3 is devoted to the notion of symplectic relation within the category of the symplectic manifolds. In Chaps. 4 and 5 we specialize our analysis within the category of the cotangent bundles. Our analysis is based on the notion of a generating family of a Lagrangian set, which is an extension of that of a generating family of a Lagrangian submanifold (or of a symplectic relation). This extension turns out to be necessary in dealing with the composition of symplectic relations.

Indeed, if the composition of two smooth symplectic relations, which are submanifolds of Cartesian products, no longer yields a smooth relation, then we can replace the composition of the relations with the composition of their generating families, which are always smooth objects.

The symplectic formulation of Hamiltonian optics, presented in Chaps. 6 and 7, is based on the fact that, from a geometrical viewpoint, a Hamilton–Jacobi equation is a coisotropic submanifold of a cotangent bundle and that a *geometrical solution* is a Lagrangian set contained in it. The solutions of a Hamilton–Jacobi equation are then described by generating families, and not by an "ordinary" function as in the classical theory.

There are two fundamental symplectic relations associated with a Hamilton–Jacobi equation, the *characteristic relation* and the *characteristic reduction*. The two corresponding generating families are called *Hamilton principal functions* and *complete solutions*.

The characteristic relation is a singular Lagrangian submanifold, thus the Hamilton principal function is necessarily a generating family and not a two-point function as it appears in the classical theory. Furthermore, Cauchy data (or *sources* of systems of rays), mirrors, and lenses are represented by symplectic relations, thus by generating families. Then the Cauchy problem and the action of a lens or of a mirror on a system of rays are translated into the composition of symplectic relations or of generating families.

In Chap. 5 it is shown that the use of generating families cannot be avoided if we want to give a global meaning to the Hamilton characteristic function, from which all solutions of the Hamilton–Jacobi equation can be derived, or if we want to describe very singular optical phenomena.

Symplectic relations and generating families can also play an interesting role in the control theory of static systems, including thermostatic systems. Chapter 8 is devoted to this matter. Our approach is based on the notion of *control relation* and on an extended version of the *virtual work principle* for constrained systems with noncontrolled degrees of freedom (*hidden variables*). Several examples of singular phenomena concerning static and thermostatic systems are illustrated. In particular, it is shown that the *Maxwell rule of equal areas* is a theorem following, through pure mathematical reasoning, from the extended virtual work principle. Thermostatics of simple and composite systems are described here in the four-dimensional state space, with global coordinates (S, V, P, T), entropy, volume, pressure, and absolute temperature, endowed with the natural symplectic structure induced by the first principle of thermodynamics.

Supplementary topics are illustrated in Chap. 9. Chapter 10 is devoted to the calculus of global Hamilton principal functions for the eikonal equations on the two-dimensional sphere $\mathbb{S}_2$ and pseudo-sphere $\mathbb{H}_2$.

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Sergio Benenti

Torino, June 2011

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This book is an enhanced version of a first edition in Russian.

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Oratio, Sermones, VI, 1

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