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Descriptive Topology in Selected Topics of Functional Analysis

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To our Friend and Teacher
Prof. Dr. Manuel Valdivia

Preface

We invoke (descriptive) topology recently applied to (functional) analysis of infinite-dimensional topological vector spaces, including Fréchet spaces, (*LF*)-spaces and their duals, Banach spaces $C(X)$ over compact spaces X, and spaces $C_p(X)$, $C_c(X)$ of continuous real-valued functions on a completely regular Hausdorff space X endowed with pointwise and compact–open topologies, respectively. The (*LF*)-spaces and duals particularly appear in many fields of functional analysis and its applications: distribution theory, differential equations and complex analysis, to name a few.

Our material, much of it in book form for the first time, carries forward the rich legacy of Köthe's *Topologische lineare Räume* (1960), Jarchow's *Locally Convex Spaces* (1981), Valdivia's *Topics in Locally Convex Spaces* (1982), and Pérez Carreras and Bonet's *Barrelled Locally Convex Spaces* (1987). We assume their (standard English) terminology. A topological vector space (tvs) must be Hausdorff and have a real or complex scalar field. A *locally convex space* (lcs) is a tvs that is locally convex. Engelking's *General Topology* (1989) serves as a default reference for general topology.

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