Thomas M. Liggett

Interacting Particle Systems

Springer-Verlag
New York Berlin Heidelberg Tokyo

Thomas M. Liggett
Department of Mathematics
University of California
Los Angeles, CA 90024
U.S.A.

AMS Subject Classifications: 60-02, 60K35, 82A05

Library of Congress Cataloging in Publication Data
Liggett, Thomas M. (Thomas Milton)
Interacting particle systems.
(Grundlehren der mathematischen Wissenschaften; 276)
Bibliography: p.
Includes index.
1. Stochastic processes. 2. Mathematical physics.
3. Biomathematics. I. Title. II. Series.
QC20.7.S8L54 1985 530.1'5 84-14152

With 6 Illustrations.

Typeset by J. W. Arrowsmith Ltd., Bristol, England.

9 8 7 6 5 4 3 2 1

ISBN-13: 978-1-4613-8544-8 e-ISBN-13:978-1-4613-8542-4
DOI: 10.1007/978-1-4613-8542-4

To my family:
Chris, Tim, Amy

Preface

At what point in the development of a new field should a book be written about it? This question is seldom easy to answer. In the case of interacting particle systems, important progress continues to be made at a substantial pace. A number of problems which are nearly as old as the subject itself remain open, and new problem areas continue to arise and develop. Thus one might argue that the time is not yet ripe for a book on this subject. On the other hand, this field is now about fifteen years old. Many important problems have been solved and the analysis of several basic models is almost complete. The papers written on this subject number in the hundreds. It has become increasingly difficult for newcomers to master the proliferating literature, and for workers in allied areas to make effective use of it. Thus I have concluded that this is an appropriate time to pause and take stock of the progress made to date. It is my hope that this book will not only provide a useful account of much of this progress, but that it will also help stimulate the future vigorous development of this field.

My intention is that this book serve as a reference work on interacting particle systems, and that it be used as the basis for an advanced graduate course on this subject. The book should be of interest not only to mathematicians, but also to workers in related areas such as mathematical physics and mathematical biology. The prerequisites for reading it are solid one-year graduate courses in analysis and probability theory, at the level of Royden (1968) and Chung (1974), respectively. Material which is usually covered in these courses will be used without comment. In addition, a familiarity with a number of other types of stochastic processes will be helpful. However, references will be given when results from specialized parts of probability theory are used. No particular knowledge of statistical mechanics or mathematical biology is assumed. While this is the first book-length treatment of the subject of interacting particle systems, a number of surveys of parts of the field have appeared in recent years. Among these are Spitzer (1974a), Holley (1974a), Sullivan (1975b), Liggett (1977b), Stroock (1978), Griffeath (1979a, 1981), and Durrett (1981). These can serve as useful complements to the present work.

This book contains several new theorems, as well as many improvements on existing results. However, most of the material has appeared in one form or another in research papers. References to the relevant papers are given in the "Notes and References" section for each chapter. The bibliography contains not only the papers which are referred to in those sections, but also a fairly complete list of papers on this general subject. In order to encourage further work, I have listed a total of over sixty open problems at the end of the appropriate chapters. It should be understood that these problems are not all of comparable difficulty or importance. Undoubtedly, some will have been solved by the time this book is published.

The following remarks should help the reader orient himself to the book. Some of the most important models in the subject are described in the Introduction. The main questions involving them and a few of the most interesting results about them are discussed there as well. The treatment here is free of the technical details which become necessary later, so this is certainly the place to start reading the book.

The first chapter deals primarily with the problem of existence and uniqueness for interacting particle systems. In addition, it contains (in Section 4) several substantive results which follow from the construction and are rather insensitive to the precise nature of the interaction. From a logical point of view, the construction of the process must precede its analysis. However, the construction is more technical, and probably less interesting, than the material in the rest of the book. Thus it is important not to get bogged down in this first chapter. My suggestion is that, on the first reading, one concentrate on the first four sections of Chapter I, and perhaps not spend much time on the proofs there. Little will be lost if in later chapters one is willing to assume that the global dynamics of the process are uniquely determined by the informal infinitesimal description which is given. The martingale formulation which is presented following Section 4 has played an important role in the development of the subject, but will be used only occasionally in the remainder of this book.

Many of the tools which are used in the study of interacting particle systems are different from those used in other branches of probability theory, or if the same, they are often used differently. The second chapter is intended to introduce the reader to some of these tools, the most important of which are coupling and duality. In this chapter, the use of these techniques is illustrated almost exclusively in the context of countable state Markov chains, in order to facilitate their mastery. In addition, the opportunity is taken there to prove several nonstandard Markov chain results which are needed later in the book.

In Chapter III, the ideas and results of the first two chapters are applied to general spin systems—those in which only one coordinate changes at a time. It is here, for example, that the general theory of attractive systems is developed, and that duality and the graphical representation are introduced. Chapters IV–IX treat specific types of models: the stochastic Ising model, the voter model, the contact process, nearest-particle systems, the exclusion process, and processes with unbounded values. These chapters have been written so that they are largely independent of one another and may be read separately. A good first exposure to this book can be obtained by lightly reading the first four sections of Chapter I, reading the first half of Chapter II, Chapter III, and then any or all of Chapters IV, V, and VI.

While I have tried to incorporate many of the important ideas, techniques, results, and models which have been developed during the past fifteen years, this book is not an exhaustive account of the entire subject of interacting particle systems. For example, all models considered here have continuous time, in spite of the fact that a lot of work has been done on analogous discrete time systems, particularly in the Soviet Union. Not treated at all or barely touched on are important advances in the following closely related subjects: infinite systems of stochastic differential equations (see, for example, Holley and Stroock (1981), Shiga (1980a, b) and Shiga and Shimizu (1980)), measure-valued diffusions (see, for example, Dawson (1977) and Dawson and Hochberg (1979, 1982)), shape theory for finite interacting systems (see, for example, Richardson (1973), Bramson and Griffeath (1980c, 1981), Durrett and Liggett (1981), and Durrett and Griffeath (1982)), renormalization theory for interacting particle systems (see, for example, Bramson and Griffeath (1979b) and Holley and Stroock (1978b, 1979a)), cluster processes (see, for example, Kallenberg (1977), Fleischmann, Liemant, and Matthes (1982), and Matthes, Kerstan, and Mecke (1978)), and percolation theory (see, for example, Kesten (1982) and Smythe and Wierman (1978)).

The development of the theory of interacting particle systems is the result of the efforts and contributions of a large number of mathematicians. There are many who could be listed here, but if I tried to list them, I would not know where to stop. In any case, their names appear in the "Notes and References" sections, as well as in the Bibliography. I would particularly like to single out Rick Durrett, David Griffeath, Dick Holley, Ted Harris, and Frank Spitzer, both for their contributions to the subject and for the influence they have had on me. Enrique Andjel, Rick Durrett, David Griffeath, Dick Holley, Claude Kipnis, and Tokuzo Shiga have read parts of this book, and have made valuable comments and found errors in the original manuscript.

Since this is my first book, this is a good place to acknowledge the influence which Sam Goldberg at Oberlin College, and Kai Lai Chung and Sam Karlin at Stanford University had on my first years as a probabilist. I would like to thank Chuck Stone for his encouragement during the early years of my work on interacting particle systems, and in particular for handing me a preprint of Spitzer's 1970 paper with the comment that I would probably find something of interest in it. This book is proof that he was right.

More than anyone else, it was my wife, Chris, who convinced me that I should write this book. In addition to her moral support, she contributed greatly to the project through her excellent typing of the manuscript. Finally, I would like to acknowledge the financial support of the National Science Foundation, both during the many years I have spent working on this subject, and particularly during the past two years in which I have been heavily involved in this writing project.

Contents

Frequently Used Notation

S	A finite or countable set of sites.
Z^d	The d-dimensional integer lattice.
Y	The collection of finite subsets of S or $S \cup \{\infty\}$.
X	The state space of the process; usually $\{0, 1\}^S$.
$C(X)$	The continuous functions on X.
$D(X)$	The Lipschitz functions on X (see Section 3 of Chapter I).
$\mathcal{M}$	The increasing continuous functions on X.
$\mathcal{D}$	The functions on X which depend on finitely many coordinates.
$\mathcal{P}$	The probability measures on X.
$\mathcal{S}$	When $S = Z^d$, the translation invariant elements of $\mathcal{P}$.
$\mathcal{S}_e$	The extremal (or ergodic) elements of $\mathcal{S}$.
$\mathcal{I}$	The elements of $\mathcal{P}$ which are invariant for the process.
$\mathcal{I}_e$	The extremal elements of $\mathcal{I}$.
$\mathcal{R}$	The elements of $\mathcal{P}$ which are reversible for the process.
$\mathcal{G}$	The Gibbs measures corresponding to some potential.
$\mathcal{G}_e$	The extremal Gibbs measures.
δ_0, δ_1	The pointmasses on $\eta \equiv 0$ and $\eta \equiv 1$.
μ or ν	Typical elements of $\mathcal{P}$.
$\mu \leq \nu$	Stochastic monotonicity (see Definition 2.1 of Chapter II).
η_t or ζ_t	The Markov process which represents an interacting particle system.
$c(x, \eta)$	The flip rate at $x \in S$ when the configuration is $\eta \in X$.
$S(t)$	The semigroup corresponding to the process.
$\mu S(t)$	The distribution at time t when the initial distribution is $\mu \in \mathcal{P}$.
Ω	The generator or pregenerator of the process.
$\mathcal{D}(\Omega)$	The domain of Ω.
$\mathcal{R}(\Omega)$	The range of Ω.
Re	The real part of a complex number.
$p^{(n)}(x, y)$	The n-step transition probabilities for a discrete time Markov chain.
$p_t(x, y)$	The transition probabilities for a continuous time Markov chain.
$\mathcal{H}$	Harmonic functions; often with some additional constraints.