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Chuanming Zong

Strange Phenomena in Convex and Discrete Geometry

Edited by James J. Dudziak

Springer

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Preface

Convex and Discrete Geometry, in the sense used by many mathematicians, is one of the most intuitive subjects in mathematics. It has the characteristic that many of its hardest problems, such as the sphere packing problem or Borsuk's problem, can be explained, along with their conjectured answers, to a layman in a few minutes. However, proofs of the conjectured answers to some of these simply stated problems often have cost the best mathematicians decades or centuries of effort. More surprisingly, some of these commonly believed conjectures, whose truth seemed intuitively certain, were not true. The conjectured answer to Borsuk's problem is an example. Furthermore, there are problems in Convex and Discrete Geometry whose answers are so counterintuitive and strange that they can hardly be believed before reading their proofs. The purpose of this book is to present just some of the most famous problems in Convex and Discrete Geometry which possess such incredible answers.

Although this book deals with difficult problems and presents some of the most recent advances in Convex and Discrete Geometry, it is self-contained and can be understood by any trained mathematician.

For invaluable help and consultation, I am obliged to Professors M. Berger, J. Dudziak, P.M. Gruber, E. Hlawka, D. Larman, P. Mani-Levitska, R. Pollack, R. Schneider, Y. Wang, J.M. Wills, and G. Xu. Professor J. Dudziak's thorough editorial work has improved the quality of the final manuscript. Nevertheless, the responsibility is wholly mine for any mistakes and faults of exposition that still remain in the book. The staff at Springer-Verlag in New York have been courteous, competent, and helpful, especially Mr. T. von Foerster and Ms. J. Wolkowicki. This work was supported by the National Scientific Foundation of China. Its final revision was done while I was a guest at IHES, ETH Zürich, and UCL. I am very grateful to these institutions for their support and hospitality.

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Basic Notation

For the convenience of the reader, we list here the following notation, which will be used throughout this book:

R^n: Euclidean space of n-dimensions.

$\mathbb{Z}$: The set of all integers.

x or x_i: Points in R^n.

x^j: The j-th coordinate of x.

conv$\{X\}$: The convex hull of a given set X. Thus,

$$\text{conv}\{X\} = \left\{\sum \lambda_i x_i : \text{each } x_i \in X, \text{ each } \lambda_i \geq 0, \text{ and } \sum \lambda_i = 1\right\}.$$

$\langle x, y\rangle$: The inner product of x and y.

$\|x - y\|$: The Euclidean distance between x and y.

$d(X)$: The diameter of X. Thus,

$$d(X) = \sup_{x,y \in X} \|x - y\|.$$

$d(X, Y)$: The Euclidean distance between sets X and Y. Thus,

$$d(X, Y) = \min_{x \in X,\ y \in Y} \|x - y\|.$$

K: An n-dimensional convex body, i.e., a compact convex set in R^n with nonempty interior.

$\partial(K)$: The boundary of K.

int(K): The interior of K.

$v(K)$: The volume of K.

$s(K)$: The surface area of K.
$D(K)$: The difference body of K. Thus, $D(K) = \{x - y : x, y \in K\}$.
$\delta^H(K_1, K_2)$: The Hausdorff distance between K_1 and K_2. Thus,

$$\delta^H(K_1, K_2) = \max\left\{\sup_{x\in K_1}\inf_{y\in K_2}\|x-y\|,\ \sup_{y\in K_2}\inf_{x\in K_1}\|x-y\|\right\}.$$

$\mathcal{K}$: The space of all n-dimensional convex bodies with the Hausdorff metric δ^H.
B: The n-dimensional unit ball centered at the origin o.
ω_n: The volume of B. Thus,

$$\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}.$$

W: The n-dimensional unit cube $\{(x^1, x^2, \ldots, x^n) : \|x^i\| \le \frac{1}{2},\ 1 \le i \le n\}$.
S: An n-dimensional simplex.
P: An n-dimensional polytope.
C: An n-dimensional centrally symmetric convex body.
$\mathcal{C}$: The space of all n-dimensional centrally symmetric convex bodies with the Hausdorff metric δ^H.
Λ: An n-dimensional lattice.

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