Universitext

Editors

Robert B. Reisel

Elementary Theory of Metric Spaces

A Course in Constructing Mathematical Proofs

Springer-Verlag
New York Heidelberg Berlin

Robert B. Reisel
Loyola University of Chicago
Department of Mathematical Sciences
Chicago, Illinois 60626
U.S.A.

Library of Congress Cataloging in Publication Data

Reisel, Robert B.
Elementary theory of metric spaces.

(Universitext)
Includes index.
1. Functions of real variables. 2. Functional analysis. 3. Metric spaces. I. Title.
QA331.R45 514'.32 82-864
AACR2

AMS Classifications: 26-01, 46-01

9 8 7 6 5 4 3 2 1

ISBN-13: 978-0-387-90706-2 e-ISBN-13: 978-1-4613-8188-4
DOI: 10.1007/978-1-4613-8188-4

Preface

Science students have to spend much of their time learning how to do laboratory work, even if they intend to become theoretical, rather than experimental, scientists. It is important that they understand how experiments are performed and what the results mean. In science the validity of ideas is checked by experiments. If a new idea does not work in the laboratory, it must be discarded. If it does work, it is accepted, at least tentatively. In science, therefore, laboratory experiments are the touchstones for the acceptance or rejection of results.

Mathematics is different. This is not to say that experiments are not part of the subject. Numerical calculations and the examination of special and simplified cases are important in leading mathematicians to make conjectures, but the acceptance of a conjecture as a theorem only comes when a proof has been constructed. In other words, proofs are to mathematics as laboratory experiments are to science. Mathematics students must, therefore, learn to know what constitute valid proofs and how to construct them. How is this done? Like everything else, by doing. Mathematics students must try to prove results and then have their work criticized by experienced mathematicians. They must critically examine proofs, both correct and incorrect ones, and develop an appreciation of good style. They must, of course, start with easy proofs and build to more complicated ones. This is usually done in courses, like abstract algebra or real analysis, in the junior and senior years of college, but this is almost too late in the curriculum. Furthermore, with the increase in the number of computer applications studied in these and in other courses, it is becoming more difficult to find enough time for a critical study of proof techniques. This book is intended to provide a text for an earlier course that would emphasize proofs while teaching useful and important mathematics.

As I said, students learn to construct proofs by actually working out proofs. Therefore, this book is not a theoretical treatise of logic or proof theory but is an actual text on metric spaces. Instead of giving the proofs of the results, I ask the students to supply them, giving them hints where necessary. Naturally, I have given all the definitions and some indication of what the ideas mean so that the book is self-contained. The students should be "on their own" as much as possible, but an instructor should be available to help them over their difficulties and to offer constructive criticism of their efforts.

Originally, I had intended not to include any proofs in the book, but I have relented in a few cases where it seemed unreasonable to expect beginning students to be able to complete some complicated proofs. Also I have included an appendix in which I have given proofs of selected results. I have tried to include those proofs which illustrate unfamiliar techniques or involve concepts that might cause difficulty. Ideally, the students should not look at the proofs in the appendix until they have written their own proofs, or at least tried to do so. However, if there is not enough time to cover the essentials of the theory of metric spaces and also do all the proofs, the course could be accelerated by having the students read the proofs in the appendix and only try to construct proofs for the other results.

I think that the best way to use this book is in a seminar; I give some suggestions for this below. It could, however, be used in a lecture course where many of the proofs would be assigned to the students. It would be suitable as the text or as a supplementary text in courses in general topology, real analysis or advanced calculus.

Having emphasized that the goal of the book is to teach an understanding of proofs and their construction, I now say that this should be forgotten. Students should concentrate on the actual mathematical content and try to learn and understand it. As they work through the mathematics, they will be learning proof techniques in what seems to be an incidental manner. All that is necessary is that high standards be insisted on and incorrect or incomplete proofs not be accepted.

The only prerequisite for this book is that nebulous quality called "mathematical maturity". I take this to mean that the students should be at ease working with mathematical formulas and symbols and be serious about learning mathematics. Students who have completed calculus should have this maturity. The subject matter of the book is the elementary theory of metric spaces. I say "the elementary theory" because I have restricted the topics to those whose proofs can be constructed by students working primarily by themselves. Nevertheless, the book develops all the ideas of metric spaces that are needed in a course in real analysis of functions of a single variable. The book also gives the students experience with mathematics in an axiomatic setting.

The preliminary chapter - Chapter 0 - is a brief explanation of those points of logic that students must understand in order to construct proofs. At the end of this chapter I have given some carefully written proofs for a number of exercises of Chapter I so that the students can check their first attempts at constructing proofs. They should be able to go through this chapter by themselves. A course using the book would actually begin with Chapter I. The first three chapters cover the essential material that everyone should know in order to understand modern analysis, namely, the elements of set theory, particularly the concept of a mapping, the basic geometrical ideas of metric spaces, and the properties of continuous mappings. The remaining three chapters take up sequences and completeness, connectedness, and compactness. Although these last three chapters involve general metric spaces, they are slanted toward the space of real numbers, which is the most important elementary application of these ideas and the one that is most familiar to the students. I have not included any discussion of product spaces, which would allow these results to be extended to higher dimensional spaces, because I think that product spaces are studied more effectively in the context of a topology course. Moreover, inclusion of this material would involve more difficult proofs and would extend the book beyond what could be reasonably covered in a single course. Students who have completed this book should have no difficulty reading about this or other advanced topics in books on metric spaces or topology. Mathematical induction is used in a few places, so I have included an appendix that briefly explains it.

For the past fifteen years I have used versions of this book in a seminar course. The advantage of a seminar for this course is that students not only construct proofs but also have an opportunity to examine critically the proofs presented by other students, some of which are probably incorrect. I usually have about ten students in the seminar and the course meets two hours a week for a semester. Assignments are made so that each student's report takes about a half hour. The students are told to write out their work completely, not using any other books. I am available if they run into difficulties, but I limit my aid to suggestions as to how to proceed. At the start of the course I ask the students to show me their work before presenting it in the seminar so that I can catch any gross errors or make stylistic suggestions. Later in the course the students should have gained enough confidence and experience so that it is no longer necessary for them to show me their work. The students who are not making reports take notes as in any course, and if there is material in the presentations that they do not understand, they ask about it. It is important that the students be encouraged to raise objections and questions because this is the way that their critical sense is developed. I am careful not to raise objections myself unless no one in the class has caught the error. If there are mistakes, attempts should be made to correct them in the class with the other students making suggestions. If they cannot be corrected in a reasonable period of time, the work can be postponed to the next meeting. It might not be possible to cover the entire book, so I have made some suggestions in the various chapters as to what might be omitted.

I want to mention a few technical points about the book. Results are numbered consecutively in each section with a pair of numbers, the first denoting the section and the second the particular result; thus, Theorem 3.2 is the second result of Section 3. Within a chapter a reference to a result is made by citing this number. A reference to a result in another chapter is made by including the chapter number; thus, Theorem II 3.2 is the second result of Section 3 of Chapter II. Definitions are not set off in separate statements, but the word being defined is underlined. It is a pecularity of mathematical style that definitions are expressed by using the word "if" rather than the phrase "if and only if," although the latter is actually meant. I have followed this convention. Those results whose proofs can be found in the appendix are marked with an asterisk (*). Finally, the symbol "[]" is used to indicate the end of a proof.

As I mentioned above, this book has reached its present form only after a considerable period of time and experimentation. I would like to list the names of all my colleagues who have used forms of this book over the years, but I am sure that I would forget to include some of those who left Loyola University years ago. Therefore, I will just express my thanks to all my colleagues, both past and present, for their suggestions, support and encouragement. I would also like to thank the many students who suffered through the seminar as it developed into its present form. Their successes and failures helped me to learn what can be reasonably expected in a course of this kind. Lastly, I want to thank my wife for her patience, encouragement and love. She was and is indispensable.

CONTENTS