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Differential Forms

A Heuristic Introduction

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Preface

A working knowledge of differential forms so strongly illuminates the calculus and its developments that it ought not be too long delayed in the curriculum. On the other hand, the systematic treatment of differential forms requires an apparatus of topology and algebra which is heavy for beginning undergraduates. Several texts on advanced calculus using differential forms have appeared in recent years. We may cite as representative of the variety of approaches the books of Fleming [2],[1] Nickerson-Spencer-Steenrod [3], and Spivak [6]. Despite their accommodation to the innocence of their readers, these texts cannot lighten the burden of apparatus exactly because they offer a more or less full measure of the truth at some level of generality in a formally precise exposition. There is consequently a gap between texts of this type and the traditional advanced calculus. Recently, on the occasion of offering a beginning course of advanced calculus, we undertook the experiment of attempting to present the technique of differential forms with minimal apparatus and very few prerequisites. These notes are the result of that experiment.

Our exposition is intended to be heuristic and concrete. Roughly speaking, we take a differential form to be a multi-dimensional integrand, such a thing being subject to rules making change-of-variable calculations automatic. The domains of integration (manifolds) are explicitly given "surfaces" in Euclidean space. The differentiation of forms (exterior

(1) Numbers in brackets refer to the Bibliography at the end.

differentiation) is the obvious extension of the differential of functions, and this completes the apparatus. To avoid the geometric and not quite elementary subtleties of a correct proof of the general Stokes formula we offer instead a short plausibility argument which we hope will be found attractive as well as convincing. This is one of several abbreviations we have made in the interests of maintaining an elementary level of exposition.

The prerequisite for this text is a standard first course of calculus and a bit more. The latter, though not very specific, may be described as some familiarity with Euclidean space of k dimensions, with k-by-k matrices and the row-by-column rule for multiplying them, and with the simpler facts about k-by-k determinants. Serious beginning undergraduates seem generally to possess this equipment at the present time. Linear algebra proper is not required, except at one place in Chapter 6, where we must diagonalize a real symmetric matrix. For this theorem, and for several other facts of algebra (such as those mentioned above), we offer references to the text [5] of Schreier and Sperner. There the matters in question are well presented without prerequisites. For analytical matters we provide citations to Courant [1]. We have tried to design the text so that, with the books of Courant and Schreier-Sperner as his only other equipment, the industrious reader working alone will find here an essentially self-contained course of study. However, the better use of this text is probably its obvious one as part of a modern sophomore or junior course of advanced calculus.

The content of each Chapter is clear from the Table of Contents, with two exceptions: in 6.2 we give the theorem on the geometric and arithmetic means, and in 7.3 we prove the isoperimetric inequality. Our notations, all standard, are listed on page (x). The symbol n.m(k) means Formula (k) in Section n.m.

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M. Schreiber
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Notations

$a \in A$	a is a member of the set A
$\vec{x}$	a vector
$\|\vec{x}\|$	the length of $\vec{x}$
$\vec{x} \cdot \vec{y}$	the scalar product of $\vec{x}$ and $\vec{y}$
$\vec{x} \times \vec{y}$	the vector product of $\vec{x}$ and $\vec{y}$
$\binom{n}{k}$	the binomial coefficient $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^k$	Euclidean space of dimension k
$f : \mathbb{R}^n \to \mathbb{R}^m$	m-vector-valued function of an n-vector argument
$Z(\phi)$	the set of zeros of $\phi : \mathbb{R}^k \to \mathbb{R}^1$
$f \circ g$	the composition $f \circ g(x) = (f(g(x))$ of functions f and g
$\omega \wedge \tau$	the wedge product of differential forms ω and τ
Λ^r	the space of r-forms
$\mathbb{W}$	the set of vector fields
$\mathbb{S}$	the set of scalar fields
$\|T\|$	the determinant of the matrix T
${}^tT$	the transpose of the matrix T
$\mathrm{tr}\,T$	the trace of the matrix T