PART III

EXTREME VALUES IN CONTINUOUS TIME

In this part of the work we shall explore extremal and related theory for continuous parameter stationary processes. As we shall see (in Chapter 13) it is possible to obtain a satisfying general theory extending that for the sequence case, described in Chapter 3 of Part II, and based on dependence conditions closely related to those used there for sequences. In particular, a general form of the Extremal Types Theorem will be obtained for the maximum

$$M(T) = \sup\{\xi(t); 0 \leq t \leq T\},$$

where $\xi(t)$ is a stationary stochastic process satisfying appropriate regularity and dependence conditions.

Before presenting this general theory, however, we shall give a detailed development for the case of stationary normal processes, for which very many explicit extremal and related results are known. For mean-square differentiable normal processes, it is illuminating and profitable to approach extremal theory through a consideration of the properties of *upcrossings* of a high level (which are analogous to the exceedances used in the discrete case). The basic framework and resulting extremal results are described in Chapters 7 and 8, respectively.

As a result of this limit theory it is possible to show that the point process of upcrossings of a level takes on an increasingly Poisson character as the level becomes higher. This and more sophisticated Poisson properties are discussed in Chapter 9, and are analogous to the corresponding results for exceedances by stationary normal sequences, given in Chapter 5.

The Poisson results provide asymptotic joint distributions for the locations and heights of any given number of the largest local maxima.

The local behaviour of a stationary normal process near a high-level upcrossing is discussed in Chapter 10, using, in particular, a simple process (the "Slepian model process") to describe the sample paths at such an upcrossing. As an interesting corollary it is possible to obtain the limiting distribution for the heights of excursions by stationary normal processes above a high level, under appropriate conditions.

In Chapter 11 we consider the joint asymptotic behaviour of the maximum and minimum of a stationary normal process, and of maxima of two or more dependent processes. In particular it is shown that—short of perfect correlation between the processes—such maxima are asymptotically independent.

While the mean square differentiable stationary normal processes form a substantial class, there are important stationary normal processes (such as the Ornstein–Uhlenbeck process) which do not possess this property. Many of these have covariance functions of the form $r(\tau) = 1 - C|\tau|^\alpha + o(|\tau|^\alpha)$ as $\tau \to 0$ for some α, $0 < \alpha < 2$ (the case $\alpha = 2$ corresponds to the mean-square differentiable processes). The extremal theory for these processes is developed in Chapter 12, using more sophisticated methods than those of Chapter 8, for which simple considerations involving upcrossings sufficed.

Finally, Chapter 13 contains the promised general extremal theory (including the Extremal Types Theorem) for stationary continuous-time processes which are not necessarily normal. This theory essentially relies on the discrete parameter results of Part II, by means of the simple device of expressing the maximum of a continuous parameter process in say time $T = n$, an integer, as the maximum of n "submaxima", over fixed intervals, viz.

$$M(n) = \max(\zeta_1, \zeta_2, \ldots, \zeta_n),$$

where $\zeta_i = \sup\{\xi(t); i - 1 \leq t \leq i\}$. It should be noted (as shown in Chapter 13) that the results for stationary normal processes given in Chapters 8 and 12 can be obtained from those in Chapter 13 by specialization. However, since most of the effort required in Chapters 8 and 12 is still needed to verify the general conditions of Chapter 13, and the normal case is particularly important, we have felt it desirable and helpful to first treat normal cases separately.