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Morton L. Curtis

# Matrix Groups

Second Edition



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To my teacher and friend  
**Raymond Louis Wilder**  
this book is affectionately dedicated.

# Introduction

These notes were developed from a course taught at Rice University in the spring of 1976 and again at the University of Hawaii in the spring of 1977. It is assumed that the students know some linear algebra and a little about differentiation of vector-valued functions. The idea is to introduce students to some of the concepts of Lie group theory-- all done at the concrete level of matrix groups. As much as we could, we motivated developments as a means of deciding when two matrix groups (with different definitions) are isomorphic.

In Chapter I "group" is defined and examples are given; homomorphism and isomorphism are defined. For a field  $k$ ,  $M_n(k)$  denotes the algebra of  $n \times n$  matrices over  $k$ . We recall that  $A \in M_n(k)$  has an inverse if and only if  $\det A \neq 0$ , and define the general linear group  $GL(n,k)$ . We construct the skew-field  $\mathbb{H}$  of quaternions and note that for  $A \in M_n(\mathbb{H})$  to operate linearly on  $\mathbb{H}^n$  we must operate on the right (since we multiply a vector by a scalar on the left). So we use row vectors for  $\mathbb{R}^n$ ,  $\mathbb{C}^n$ ,  $\mathbb{H}^n$  and write  $xA$  for the row vector obtained by matrix multiplication. We get a complex-valued determinant function on  $M_n(\mathbb{H})$  such that  $\det A \neq 0$  guarantees that  $A$  has an inverse.

Chapter II introduces conjugation on  $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$  and then an inner product  $\langle \cdot, \cdot \rangle$ . Basic properties of  $\langle \cdot, \cdot \rangle$  are given and then for  $k \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$  we define the orthogonal group

$$\mathfrak{O}(n,k) = \{A \in M_n(k) \mid \langle xA, yA \rangle = \langle x, y \rangle \text{ for all } x, y \in k^n\} .$$

$\mathfrak{O}(n, \mathbb{R})$  is written  $\mathfrak{O}(n)$  and called the orthogonal group.  $\mathfrak{O}(n, \mathbb{C})$  is written  $U(n)$  and called the unitary group.  $\mathfrak{O}(n, \mathbb{H})$  is written  $Sp(n)$  and called the symplectic group. If  $A \in \mathfrak{O}(n)$  then  $\det A \in \{1, -1\}$  and the subgroup with  $\det = 1$  is denoted by  $SO(n)$  and called the special orthogonal group. If  $A \in U(n)$  then  $\det A$  is a complex number of unit length. The subgroup with  $\det = 1$  is denoted by  $SU(n)$  and called the special unitary group. As a first example of a matrix group isomorphism we show that  $Sp(1) \cong SU(2)$ .

In Chapter III we define the first invariant (i.e., something unchanged by an isomorphism) of a matrix group; namely, its dimension. A tangent vector to a matrix group  $G$  is  $\gamma'(0)$  for some differentiable curve  $\gamma$  in  $G$  with  $\gamma(0) = I$ . The set  $T_G$  of all tangent vectors is shown to be a vector space, a real subspace of  $M_n(k)$  ( $k \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ ). The dimension of  $T_G$  (as a real vector space) is the dimension of  $G$ . Smooth homomorphisms are defined and shown to induce linear maps of tangent spaces. Then dimension is seen to be an invariant.

In order, in Chapter IV, to calculate the dimensions of our matrix groups we develop the exponential map  $\exp: M_n(k) \rightarrow GL(n, k)$  and the logarithm,  $\log: U \rightarrow M_n(k)$  where  $U$  is some neighborhood of  $I$  in  $GL(n, k)$ . We have that  $\exp$  and  $\log$  are inverses,  $\exp: V \rightarrow U$  and  $\log: U \rightarrow V$  where  $V$  is a neighborhood of  $0$  on  $M_n(k)$  and  $U$  is a neighborhood of  $I$  in  $M_n(k)$  (actually  $U \subset GL(n, k)$ ). One-parameter subgroups are defined and proved to be determined by their derivatives at  $0$ . It follows that  $T_G$  can be taken to be all derivatives of one-parameter subgroups. Lie algebras are defined and

we see that each  $T_G$  is a Lie algebra. Finally, we then calculate the dimensions of  $SO(n)$ ,  $U(n)$ ,  $SU(n)$  and  $Sp(n)$ .

In Chapter V we consider the very specific question of whether  $Sp(1)$  and  $SO(3)$  are isomorphic. We get a surjective homomorphism  $\phi: Sp(1) \rightarrow SO(3)$  with kernel  $= \{1, -1\}$ . Then we define the center of a group, show it is an invariant and then calculate  $\text{Center } Sp(1) = \{1, -1\}$  and  $\text{Center } SO(3) = \{I\}$ , proving that  $Sp(1) \neq SO(3)$ . We define quotient groups and then note that we get new groups  $\frac{G}{\text{center}}$  whenever a matrix group has nontrivial center.

In Chapter VI we do some topology which is needed in other parts of the text. All of our matrix groups are in some euclidean space and we just do topology of subsets of euclidean spaces. We give some basic results about continuity of functions, connected sets and compact sets. The proof that continuous functions preserve compactness relegated to an appendix. We consider countable bases for open sets since this is needed later in our study of maximal tori in matrix groups. Finally, there is a short section on manifolds.

Chapters VII, VIII, and IX are devoted to studying maximal tori in our matrix groups. We describe certain specific maximal tori. We prove that any two maximal tori are conjugate and that, if  $G$  is connected, then these conjugates cover  $G$ . At this stage we then know the dimension, center and rank of all of our matrix groups and these suffice to settle our original question as to which of these groups are isomorphic. At the end of Chapter IX we discuss simple groups and covering groups. The only new groups which arise are the double covers of  $SO(n)$  ( $n = 3, 4, \dots$ ). This leads to the question:

Is the double cover of  $SO(2n+1)$  isomorphic with  $Sp(n)$  ?

In Chapter X we construct the double cover  $Spin(n)$  of  $SO(n)$  using Clifford algebras. We show that  $Spin(1) \cong S^0$ ,  $Spin(2) \cong S^1$ ,  $Spin(3) \cong Sp(1)$  ( $= S^3$ ) and  $Spin(4) \cong Sp(1) \times Sp(1)$ . Finally we show that

$$Spin(5) \cong Sp(2) \quad \text{and} \quad Spin(6) \cong SU(4) .$$

In Chapter XI we finish our job by showing that  $Sp(n) \not\cong Spin(2n+1)$  for any  $n > 2$ . This is done by looking at normalizers of maximal tori and resulting Weyl groups. If the normalizer is the semidirect product of the torus and the Weyl group, we say that the normalizer splits. If  $Spin(2n+1)$  and  $Sp(n)$  were isomorphic we would have

$$\frac{Spin(2n+1)}{\text{center}} \cong \frac{Sp(n)}{\text{center}} .$$

So our result is a consequence of the following three results.

- (\*) The normalizer in  $Sp(n)$  does not split for any  $n$ .
- (+) The normalizer in  $\frac{Sp(n)}{\text{center}}$  splits  $\Leftrightarrow n \in \{1,2\}$ .
- (r) The normalizer in  $\frac{Spin(2n+1)}{\text{center}}$  ( $= SO(2n+1)$ ) splits for  $n = 1,2,3,\dots$ .

Finally, in Chapter XII we give a brief introduction to abstract Lie groups.



## Introduction to the Second Edition

I have taken this occasion to add a short chapter introducing the concept of the roots of a compact connected Lie group. But mostly it has been an opportunity to correct a multitude of typos and some mathematical errors. In this regard I have been greatly helped by Bruce Sagan (University College of Wales) and by George Seligman (Yale). Both have sent me numerous helpful comments.

Morton L. Curtis

Houston, Texas

June 1984

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