Part II

Critical Points of Smooth Functions

Generic functions have only nondegenerate critical points. However in the study of families of functions the simplest nondegeneracies occur (irremovable by small perturbations). For example, the family $f(x, t) = x^3 - tx$ has a degenerate critical point for the value zero of the parameter t and every nearby family has the same degeneracy for a nearby value of the parameter. For a greater number of parameters more complicated degeneracies arise.

The problem of classifying all these degeneracies appears at first glance to be hopeless. However after the initial part of such a classification had been calculated it became apparent that this part is sufficiently simple: the classification of the simplest degeneracies turns out to be connected with the classification of the simple Lie groups, with the theory of groups generated by reflections, with the theory of braids and with the classification of the regular polyhedra in ordinary three-dimensional space.

In this Part we describe the initial steps in the classification of critical points of functions, including the classification of simple (or 0-modal), unimodal and bimodal singularities, and also the classification of all singularities of multiplicity $\mu \leqq 16$.

The number ν of classes (of stable μ-equivalence, defined below) of complex singularities of multiplicity μ is given, for $\mu \leqq 16$, by the following table:

μ	**1**	**2**	3	**4**	5	6	7	**8**	9	10	11	12	13	14	15	**16**
ν	**1**	**1**	1	**2**	2	3	3	**4**	4	7	11	15	14	17	22	**32**

The set of singularities not yet fully classified has codimension 11, hence all the critical points occurring in generic families of functions depending on no more than 10 parameters have been fully classified.

The classification of the simplest singularities is discrete but more highly degenerate singularities have moduli.

The *modality* m of a point $x \in X$ under the action of a Lie group G on a manifold X is the least number such that a sufficiently small neighbourhood of x may be covered by a finite number of m-parameter families of orbits. The point x is said to be *simple*, if its modality is 0, that is if its neighbourhood intersects only a finite number of orbits.

The *modality of a function-germ* at a critical point with critical value 0 is defined to be the modality of a sufficient jet in the space of jets of functions with critical point 0 and critical value 0.

Two germs are said to be *stably equivalent* if they become R-equivalent* after the addition of quadratic forms in an appropriate number of variables.

Theorem 1 (see [9], [10]): *Simple germs of holomorphic functions (germs with* $m = 0$) *are given, up to stable equivalence, by the following list:*

$$A_k\colon f(x) = x^{k+1},\ k \geqq 1;$$
$$D_k\colon f(x, y) = x^2y + y^{k-1},\ k \geqq 4;$$
$$E_6\colon f(x, y) = x^3 + y^4;$$
$$E_7\colon f(x, y) = x^3 + xy^3;$$
$$E_8\colon f(x, y) = x^3 + y^5.$$

The connection between these singularities and the simple Lie algebras or groups generated by reflections, denoted by the same symbols, is discussed in [11]. These singularities may also be obtained from the regular polyhedra in three-dimensional Euclidean space or more precisely from the discrete subgroups of the group SU(2): they describe relations between the basic invariants of the groups. A_k corresponds to the polygons, D_k to the dihedra (the two-sided polygons), E_6 to the tetrahedron, E_7 to the octahedron and E_8 to the icosahedron. For more details see [17].

Theorem 2 (*see* [12]): *The unimodal germs (germs with* $m = 1$) *are given, up to stable equivalence, by the triply indexed series of one-parameter families*

$$T_{p,q,r}\colon f(x, y, z) = axyz + x^p + y^q + z^r,$$

* Two functions are said to be R-equivalent if one can be turned into the other by a suitable (diffeomorphic) change of the independent variables.

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1,\ a\neq 0;$$

by the three one-parameter families of parabolic germs

$$\begin{aligned} P_8 &= T_{3,3,3}: f(x,y,z) = x^3+y^3+z^3+axyz,\ a^3+27\neq 0,\\ X_9 &= T_{2,4,4}: f(x,y,z) = x^4+y^4+z^2+ax^2y^2,\ a^2\neq 4,\\ J_{10} &= T_{2,3,6}: f(x,y,z) = x^3+y^6+z^2+ax^2y^2,\ 4a^3+27\neq 0,\end{aligned}$$

and by further 14 *exceptional one-parameter families, enumerated in the following table* (the meaning of the columns in which is described below):

Notation	Normal form	Indices of homogeneity	Coxeter numbers	Dolgachev numbers	Gabrielov numbers	Dual class
Q_{10}	$x^2z+y^3+z^4+ayz^3$	8 9 6	−24	2 3 9	3 3 4	E_{14}
Q_{11}	$x^2z+y^3+yz^3+az^5$	7 6 4	−18	2 4 7	3 3 5	Z_{13}
Q_{12}	$x^2z+y^3+z^5+ayz^4$	6 5 3	−15	3 3 6	3 3 6	Q_{12}
S_{11}	$x^2z+yz^2+y^4+ay^3z$	6 5 4	−16	2 5 6	3 4 4	W_{13}
S_{12}	$x^2z+yz^2+xy^3+ay^5$	5 4 3	−13	3 4 5	3 4 5	S_{12}
U_{12}	$x^3+y^3+z^4+axyz^2$	4 4 3	−12	4 4 4	4 4 4	U_{12}
Z_{11}	$x^3y+y^5+z^2+axy^4$	15 8 6	−30	2 3 8	2 4 5	E_{13}
Z_{12}	$x^3y+xy^4+z^2+ay^6$	11 6 4	−22	2 4 6	2 4 6	Z_{12}
Z_{13}	$x^3y+y^6+z^2+axy^5$	9 5 3	−18	3 3 5	2 4 7	Q_{11}
W_{12}	$x^4+y^5+z^2+ax^2y^3$	10 5 4	−20	2 5 5	2 5 5	W_{12}
W_{13}	$x^4+xy^4+z^2+ay^6$	8 4 3	−16	3 4 4	2 5 6	S_{11}
E_{12}	$x^3+y^7+z^2+axy^5$	21 14 6	−42	2 3 7	2 3 7	E_{12}
E_{13}	$x^3+xy^5+z^2+ay^8$	15 10 4	−30	2 4 5	2 3 8	Z_{11}
E_{14}	$x^3+y^8+z^2+axy^6$	12 8 3	−24	3 3 4	2 3 9	Q_{10}

These 14 singularities may be obtained from 14 triangles in the Lobachevskiĭ plane or more precisely from the discrete subgroups of the group SU(1, 1) determined by them. The normal form with $a = 0$ describes the unique relation between the invariants of the algebra of integral automorphic forms. Precisely for the 14 triangles this algebra has three generators; the angles of these triangles are π/(Dolgachev number).

Dolgachev, to whom this construction is due, and Pinkham have also described a method for obtaining the 14 exceptional singularities from the so-called K3 surfaces (see [54], [144]).

Theorem 3 (*see* [10], [12]): *The set of non-simple function-germs of functions of* $n \geqq 3$ *variables has codimension* 6, *while the set of germs of modality greater than* 1 *has codimension* 10 *in the space of function-germs with critical value* 0.

Therefore every s-parameter family of functions, where $s < 6$ ($s < 10$), may be put into general position by a suitable small disturbance in such a way that the germs of the functions at all the critical points become stably equivalent to germs of Theorem 1 (of Theorems 1 and 2), up to additive constants.