Introduction

A mathematically precise definition of the intuitive notion of "algorithm" was implicit in Kurt Gödel's [1931] paper on formally undecidable propositions of arithmetic. During the 1930s, in the work of such mathematicians as Alonzo Church, Stephen Kleene, Barkley Rosser and Alfred Tarski, Gödel's idea evolved into the concept of a recursive function. Church proposed the thesis, generally accepted today, that an effective algorithm is the same thing as a procedure whose output is a recursive function of the input (suitably coded as an integer). With these concepts, it became possible to prove that many familiar theories are undecidable (or non-recursive)—i.e., that there does not exist an effective algorithm (recursive function) which would allow one to determine which sentences belong to the theory. It was clear from the beginning that any theory with a rich enough mathematical content must be undecidable. On the other hand, some theories with a substantial content are decidable. Examples of such decidable theories are the theory of Boolean algebras (Tarski [1949]), the theory of Abelian groups (Szmielew [1955]), and the theories of elementary arithmetic and geometry (Tarski [1951], but Tarski discovered these results around 1930). The determination of precise lines of division between the classes of decidable and undecidable theories became an important goal of research in this area.

By an *algebra* we mean simply any structure $\langle A, f_i(i \in I)\rangle$ consisting of a nonvoid set A and a system of finitary operations f_i over A. A *variety*, or *equational class*, is a class of similar algebras defined by some set of equations. A variety is called *locally finite* if every one of its finitely generated algebras is finite. A variety is called *decidable* if and only if its first order theory is a recursive set of sentences. In this book we address the questions: Which varieties are decidable? If a variety is decidable, what can one conclude about the structure of its algebras?

W. Szmielew proved in [1955] that every axiomatically defined class of Abelian groups is decidable. Yu. L. Ershov [1972] proved that every variety of groups containing a finite non-Abelian group is undecidable. A. P. Zamyatin, in a series of papers published between 1973 and 1978, showed that every non-Abelian variety of groups is undecidable, and went on to characterize all the decidable varieties of rings and semigroups. (See the bibliography.) Inspired by Zamyatin's success, S. Burris and R. McKenzie [1981] considered varieties that were completely unrestricted in nature, except that they had to be locally finite and congruence-modular. Using Zamyatin's methods and some new techniques, they reduced the problem of determining the decidable varieties, within this domain, to two much more restricted problems: Which varieties of modules over finite rings are decidable, and which discriminator varieties are decidable?

In the present work, we remove the hypothesis of congruence-modularity from these results. We prove that a decidable locally finite variety is the product of a decidable, congruence-modular variety, and a decidable, strongly Abelian variety; and we establish a simple criterion for the decidability of a locally finite, strongly Abelian variety. We find that there are just three kinds of indecomposable, decidable, locally finite varieties, characterized by extremely different structural features of their algebras.

Varieties of the first kind are *strongly Abelian*. A decidable variety of this kind is definitionally equivalent with a class of k-sorted multi-unary algebras for some integer k. Valeriote [1986] found a simple necessary and sufficient condition for the decidability of a locally finite, strongly Abelian variety, which is reproduced here in Chapters 11 and 12.

Varieties of the second kind are *affine*. Each variety of this kind is equivalent, in a very strong fashion, to the variety of modules over some finite ring. The problem of determining which locally finite affine varieties are decidable is equivalent to that of determining for which finite rings the variety of modules is decidable—an unsolved problem. Some partial results on this problem are mentioned in Chapter 14.

Varieties of the third kind are *discriminator varieties*. The generic discriminator variety is the (decidable) class of Boolean algebras. The problem of characterizing the decidable locally finite discriminator varieties is open. However, it is known that each discriminator variety which, like the variety of Boolean algebras, is generated by a finite algebra with finitely many basic operations, is decidable.

This whole book consists of a proof of one result, Theorem 13.10. The theorem asserts that every decidable locally finite variety $\mathcal{V}$ is the product (or join) of independent decidable subvarieties, $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$, of the three kinds described above. The independence of these subvarieties means that every algebra in $\mathcal{V}$ is uniquely expressible as the direct product of three algebras, one from each of the subvarieties. The proof shows that if a locally finite variety does not decompose in this way, as the join of independent varieties of the three special kinds, then at least one of twenty different interpretations will interpret the class of all graphs into the variety. Fifteen of these interpretations are developed in this book, while the other five can be found in texts to which we refer.

The research leading up to this result was motivated only in part by a desire to know which varieties are decidable. As algebraists, we have a compulsion to reveal and describe the structural features of algebras. We hoped that the study of decidability might reveal some rather precise division of the family of all varieties into a small number of subclasses composed, on the one end, of varieties in which structure is manageable in all its aspects, and can be described, and on the other, of varieties that sustain structures of arbitrary complexity. In fact, we view our chief result, Theorem 13.10, as an important step in this direction; and it seems to indicate that the connection between decidability and manageable structure is a very close one.

The book begins with a preliminary chapter in which we introduce the concepts with which we shall be working, and the tools that we shall use. Next, the long argument that proves Theorem 13.10 is presented in Chapters 1–13. The structure of the argument and the plan of these chapters are outlined in the first paragraphs of Chapter 1. In this work, the modified Boolean powers introduced in Burris, McKenzie [1981] continue to play a substantial part; while a new ingredient, the tame congruence theory of D. Hobby, R. McKenzie [1988], has an indispensable role.

As a corollary of Theorem 13.10, we have an algorithm which leads from any finite algebra with finitely many basic operations to a finite ring with unit, so that the variety generated by the algebra is decidable if and only if the variety of unitary left modules over the ring is decidable. A second corollary is that any finite algebra contained in a decidable variety generates itself a variety that is definitionally equivalent with a finitely axiomatizable, decidable variety. These corollaries, and four open problems, are discussed in Chapter 14.

This book is divided into three parts. All of the results in Part II, as well as Lemma 13.12, are drawn from Valeriote's [1986] doctoral dissertation. The authors are grateful to Stanley Burris and Ross Willard for a very careful and critical reading of the manuscript. The improvement owing to their suggestions is very visible to us, especially in Chapters 2 and 6. We would also like to thank Bradd Hart for the helpful suggestions he provided on the presentation of Chapter 11.