

# **Applied Mathematical Sciences**

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(continued following index)

Carlo Marchioro    Mario Pulvirenti

# Mathematical Theory of Incompressible Nonviscous Fluids

With 85 Illustrations



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# Preface

Fluid dynamics is an ancient science incredibly alive today. Modern technology and new needs require a deeper knowledge of the behavior of real fluids, and new discoveries or steps forward pose, quite often, challenging and difficult new mathematical problems. In this framework, a special role is played by incompressible nonviscous (sometimes called *perfect*) flows. This is a mathematical model consisting essentially of an evolution equation (the Euler equation) for the velocity field of fluids. Such an equation, which is nothing other than the Newton laws plus some additional structural hypotheses, was discovered by Euler in 1755, and although it is more than two centuries old, many fundamental questions concerning its solutions are still open. In particular, it is not known whether the solutions, for reasonably general initial conditions, develop singularities in a finite time, and very little is known about the long-term behavior of smooth solutions. These and other basic problems are still open, and this is one of the reasons why the mathematical theory of perfect flows is far from being completed.

Incompressible flows have been attacked, by many distinguished mathematicians, with a large variety of mathematical techniques so that, today, this field constitutes a very rich and stimulating part of applied mathematics. The idea of writing the present book was motivated by the fact that, although there are many interesting books on the subject, no recent one, to our knowledge, is oriented toward mathematical physics. By this we mean a book that is mathematically rigorous and as complete as possible without hiding the underlying physical ideas, presenting the arguments in a natural order, from basic questions to more sophisticated ones, proving everything and trying, at the same time, to avoid boring technicalities. This is our purpose.

The book does not require a deep mathematical knowledge. The required

background is a good understanding of the classical arguments of mathematical analysis, including the basic elements of ordinary and partial differential equations, measure theory and analytic functions, and a few notions of potential theory and functional analysis.

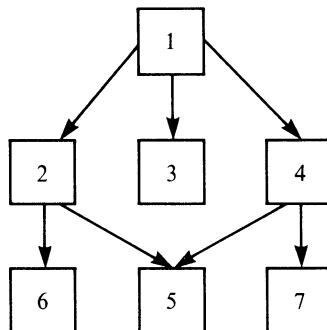
The exposition is as self-contained as possible. Several appendices, devoted to technical or elementary classical arguments, are included. This does not mean, however, that the book is easy to read. In fact, even if we tried to present the topics in an elementary fashion and in the simplest cases, the style is, in general, purely mathematical and rather concise, so that the reader quite often is requested to spend some time in independent thinking during the most delicate steps of the exposition. Some exercises, with a varying degree of difficulty (the most difficult are marked by \*), are presented at the end of many chapters. We believe solving them is the best test to see whether the basic notions have been understood.

The choice of arguments is classical and in a sense obligatory. The presentation of the material, the relative weight of the various arguments, and the general style reflect the tastes of the authors and their knowledge. It cannot be otherwise.

The material is organized as follows: In Chapter 1 we present the basic equations of motion of incompressible nonviscous fluids (the Euler equation) and their elementary properties. In Chapter 2 we discuss the construction of the solutions of the Cauchy problem for the Euler equation. In Chapter 3 we study the stability properties of stationary solutions. In Chapter 4 we introduce and discuss the vortex model. In Chapter 5 we briefly analyze the approximation schemes for the solutions of fluid dynamical equations. Chapter 6 is devoted to the time evolution of discontinuities such as the vortex sheets or the water waves. Finally, in Chapter 7 we discuss turbulent motions. This last chapter mostly contains arguments of current research and is essentially discursive.

The final section of each chapter is generally devoted to a discussion of the existing literature and further developments. We hope that this will stimulate the reader to study and research further.

The book can be read following the natural order of the chapters, but also along the following paths:



A possible criticism of the book is that two-dimensional flows are treated in much more detail than three-dimensional ones, which are, physically speaking, much more interesting. Unfortunately, for a mathematical treatise, it cannot be otherwise: The mathematical theory of a genuine three-dimensional flow is, at present, still poor compared with the rather rich analysis of the two-dimensional case to which we address many efforts.

It is a pleasure to thank D. Benedetto, E. Caglioti, A.J. Chorin, P. Drazin, R. Esposito, T. Kato, D. Levi, R. Robert, and R. Temam for useful suggestions and, particularly, P. Laurence and C. Maffei for their constructive criticism in reading some parts of the present book. We are also grateful to H. Aref for having sent us the MacVortex program. We finally thank C. Vaughn for her advice in improving our English.

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